Anelastic (poroviscoelastic) medium – the S_H – wave problem

P.F. Daley

ABSTRACT

When considering the problem of extending seismic wave propagation in an elastic medium to a poroviscoelastic medium, replacing real quantities by complex equivalents has been the accepted way to proceed. Given the number of works dealing with, what could be called the inadequacy of this method of this approach, another line of reasoning might be in order. Starting with Biot's equations for a poroviscoelastic medium, employing a simplification route, results in the S_H (modified) potential related to the vector equation of motion. Biot's theoretical development of wave propagation in a medium comprised of a fluid within a porous solid may be overly complicated for the pursuit of an alternate methodology for addressing this problem in its most basic form. As a consequence, the *telegraph* equation might be a more modest, yet informative analogue to consider, as it is a well studied problem from mathematical and physical perspectives. In what follows an S_H potential wave equation is considered with attenuation introduced in a manner similar to that inherent in the telegraph equation. Additionally, the difficult situation (Krebes and Daley, 2007) will again be revisited, as it might be rationalized that a 1 – 2% modification of a real quantity such as velocity produces imperceptible effects in, say a reflection coefficient, while the same amount of perturbation introduced to make velocity a complex quantity results in significant dissimilarities between nearly similar initial input data. This is difficult to comprehend and seemingly at least as problematic to explain.

INTRODUCTION

It might be superfluous to begin with the phrase "In the recent literature the topic of wave propagation in poroviscoelastic media has received much attention." as it has been an ongoing area of investigation for a number of decades. A series of papers by Morozov (2009a, 2009b, 2010, 2011) Morozov and Ma (2009) and others such as Lines et al. (2008), Ruud (2006) and Krebes and Daley (2007) have considered this problem. There are a significant number of other relevant citations within all of the above including the standard texts of Aki and Richards (1980, 2002) and Carcione (2007). Some of the above will be specifically referred to here.

As mentioned in the Abstract an analogy of the telegraph equation can be used to pursue a solution method. The reason for this is to not have to assume that the correspondence principle, Carcione (2007), is valid, which results it not having to require that in going from elastic to (poro)viscoelastic media does not require that real media parameters be replaced by complex valued equivalents. The telegraph equation is a problem encountered in almost all advanced undergraduate mathematical or mathematical physics texts. One of its forms for an infinite line is

$$\frac{\partial^2 u}{dx^2} - \alpha \frac{\partial u}{dt} - \gamma \frac{\partial^2 u}{dt^2} = 0\,, \quad t > 0\,. \tag{1}$$

$$u(x,t)\big|_{t=0} = \frac{\partial u}{dt}(x,t)\bigg|_{t=0} = 0\,, \quad u(0,t) = f(t)\,, \quad u(x,t)\big|_{x\to\infty} = 0\,. \qquad (2)^1$$

This problem is often used as an example in presenting Laplace transform theory (Hildebrand,1962) and can be solved using the Laplace transform tables on pp. 1020-1029 in Abramowitz and Stegun (1980). The two constants, α and γ are composed of the quantities R – resistance , C – capacitance and L – inductance (all per unit length) which are real positive quantities and $\alpha = RC$, $\gamma = LC$. As mentioned, a similar higher spatial dimension form of this equation type, specific to seismic wave propagation, may be obtained from Biot's equations (Biot (1956a, 1956b, 1956c)[2] for wave propagation in a poroviscoelastic medium for the S_H potential wave equation, or equivalently the telegraph equation may be generalized to more spatial dimensions, as

$$\mu\nabla^2\psi - b\frac{\partial\psi}{dt} - \rho\frac{\partial^2\psi}{dt^2} = F(\mathbf{x},t) = \delta(\mathbf{x})f(t)\,. \tag{3}$$

with zero initial conditions

$$\psi(\mathbf{x},t)\big|_{t=0} = \frac{\partial\psi}{dt}(\mathbf{x},t)\bigg|_{t=0} = 0\,. \tag{4}$$

where $\psi(\mathbf{x},t)$ is the S_H wave potential whose corresponding polarization vector is oriented perpendicular to the plane of incidence in what has been assumed to be a homogeneous medium. The quantity μ is Lamé's rigidity parameter, ρ — volume density and b – a dimensionally correct constant. In Biot's theory, the quantity b is defined as the mobility ratio in terms of real parameters as the mobility ratio $b = \phi^2\eta/k$, where ϕ – porosity , η – viscosity and k – permeability . All of the preceding values are real and positive quantities. What has been assumed here is that only medium types that display attenuation visible on seismic records are those composed of a solid matrix with a fluid of any type occupying the porous part of the medium. This would exclude metamorphic formations such as a serpentine layer imbedded in a basalt structure, as there is little or no attenuation associated with this (in the strictest theoretical or geological sense).

[1] Most often the boundary condition is given at some finite length ℓ as $u(x,t)\big|_{x\to\ell} = 0$ (line open) and then let $\ell \to \infty$. $du(x,t)/dx\big|_{x\to\ell} = 0$ (line grounded).

[2] Frenkel, circa 1935, published a work in Russian on this topic that has subsequently been translated to English. This work has some inconsistencies. Most present day citations in this area of research refer to Biot.

PLANE WAVE THEORY

It is probably useful to first consider a two dimensional plane wave solution for equation (3) of the form

$$\psi\left(x,z,t\right)=A\exp\left[-i\omega t+i\omega px+i\omega qz\right]. \tag{5}$$

for some nonzero amplitude A. Introducing (5) into the source free form of (3) results in

$$\left[\mu\left(i\omega p\right)^{2}+\mu\left(i\omega q\right)^{2}+i\omega b-\rho\left(i\omega\right)^{2}\right]A=0. \tag{6}$$

or as $A\neq 0$, then

$$\begin{aligned} p^{2}+q^{2}-\frac{ib}{\omega\rho\beta^{2}}-\frac{1}{\beta^{2}}&=0 \\ q^{2}=\frac{1}{\beta^{2}}\left(1+\frac{ib}{\omega\rho}\right)-p^{2} \end{aligned}. \tag{7}$$

from which it follows that

$$q=\left[\frac{1}{\beta^{2}}\left(1+\frac{i}{\omega\rho/b}\right)-p^{2}\right]^{1/2}. \tag{8}$$ [3]

Upon comparison of (8) with the vertical slowness defined in other works indicates that the dimensionless attenuation or quality factor Q is defined as $Q=\omega\rho/b$, and $p=p_r+ip_i$, with p_r and p_i being real positive quantities so that p is required to lie in the first quadrant of the complex $p-$plane to satisfy radiation conditions.

SH PLANE WAVE REFLECTION AND TRANSMISSION COEFFICIENTS

Before considering the saddle point method related to this problem it is useful to begin with the plane wave reflection and transmission coefficients at an interface between two poroviscoelastic media. Consider two $\left(1\rightarrow upper \text{ and } 2\rightarrow lower\right)$ poroviscoelastic media separated by an interface in the $\left(x,z\right)$ plane at $z=0$ with z chosen to be positive downwards. Media parameters are $\beta_k^2=\mu_k/\rho_k$, b_k, and Q_k. The incident, reflected and transmitted plane waves may be written as

[3] ω is only here as a mathematical convenience in order to make the quantity dimensionless. If one starts from a finite difference solution $\omega\rightarrow\left(\Delta t/2\right)^{-1}$. Q must be determined empirically using other methods. However $k-$effective permeability and $\eta-$effective viscosity both may be frequency dependent. Usually the ratio b is taken as the frequency dependent parameter, if this type of dependence is wanted, or $\overline{b}\rightarrow\omega b$ so that $Q=\rho/\overline{b}$.

$$\psi_{inc} = A_{inc} \exp\left[-i\omega t + i\omega px + i\omega q_1 z\right]$$
$$\psi_{ref} = A_{ref} \exp\left[-i\omega t + i\omega px - i\omega q_1 z\right]$$
$$\psi_{trn} = A_{trn} \exp\left[-i\omega t + i\omega px + i\omega q_2 z\right].$$
(9)

With $A_{ref} / A_{inc} = R_{11}^{SH}$ being the reflection coefficient and $A_{trn} / A_{inc} = R_{12}^{SH}$ the transmission coefficient the continuity of potential and potential shear stress at a plane interface between the two poroviscoelastic media using plane waves require that

$$R_{11}^{SH} - R_{12}^{SH} = -1$$
(10)

and

$$R_{11}^{SH} \mu_1 q_1 + R_{12}^{SH} \mu_2 q_2 = \mu_1 q_1 .$$
(11)

Defining D to be

$$D = \mu_1 q_1 + \mu_2 q_2$$
(12)

results in the expressions for the reflection and the transmission coefficients to be

$$R_{11}^{SH} = \frac{\rho_1 \beta_1 q_1 - \rho_2 \beta_2 q_2}{D} \quad , \quad R_{12}^{SH} = \frac{2 \rho_1 \beta_1 q_1}{D}$$
(13)

where

$$q_k = \left(\frac{1}{\beta_k^2} - p^2 \right)^{1/2} \quad \rightarrow \quad q_k = \left[\frac{1}{\beta_k^2} \left(1 + \frac{i}{Q_k} \right) - p^2 \right]^{1/2} \quad \left(k = 1,2 \right).$$
(14)

As before, $p = p_r + ip_i \left(p_r \geq 0, p_i \geq 0 \right)$.

From the paper by Morozov (2011) the medium parameters for used here for SH reflection are similar to what was used in that paper for the acoustic wave case. As Q is infinite in the upper medium, this indicates that the upper medium is elastic. As a consequence, the values of $p = p_0$ corresponding to plane wave incidence from medium *1* for the range of incident angles $\left(0 \leq \theta \leq \pi/2 \right)$ is $\left(0 \leq p_0 \leq p_1 \right)$, where p_1 is located on the real $p-$axis (Figure 1). The lower medium is assumed to be poroviscoelastic with the relation between the shear wave velocities in the two medium given by $\beta_1 < \beta_2$.

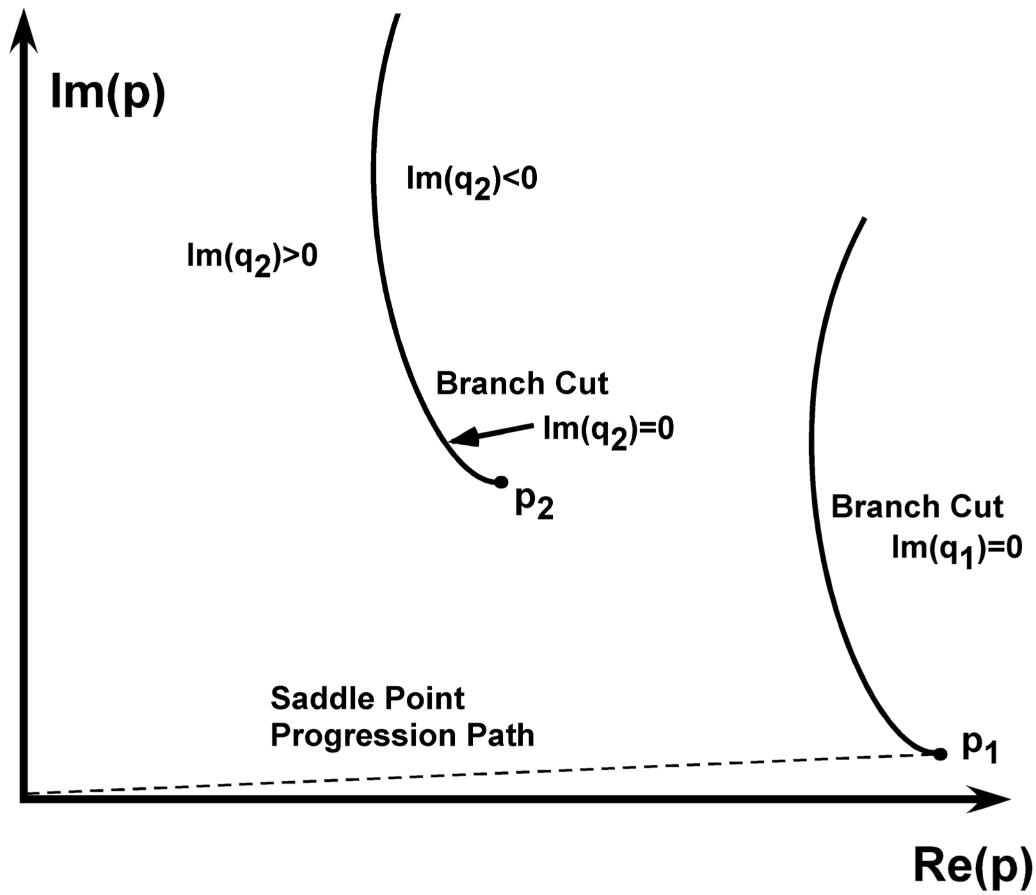

Fig. 1. This schematic shows the saddle point path for the zero – order saddle point approximation for the SH wave equation for shear wave reflection from the plane interface between the two media. The parameter values in Table 1 indicate that the saddle point path should lie along the real $p-$axis. It has been moved slightly into the first quadrant for viewing convenience.

	Shear Wave Velocity (km/s)	**Density (gm/cm^3)**	**Q**
Upper (1) Medium	1.0	1.0	∞
Lower (2) Medium	2.0	1.2	30/5

Table 1. Medium parameters taken from Morozov (2011). The upper medium is elastic.

Two plots are presented for values of $Q = 30$ and $Q = 5$ in medium 2 in Figures 2 and 3. Each of these figures consists of an upper and lower panel. The upper panel contains the amplitude plotted against the real part of p_0 while the bottom panel is the phase versus $\mathrm{Re}(p_0)$. Both the poroviscoelastic coefficients and the reference elastic case $(Q_1 = Q_2 = \infty)$ amplitudes and phases are shown in the figures. For completeness, the transmission coefficients for the two cases described in Table 1 are shown in Figures 4 and 5. In all figures, the anelastic case is plotted in blue and the elastic case is in red.

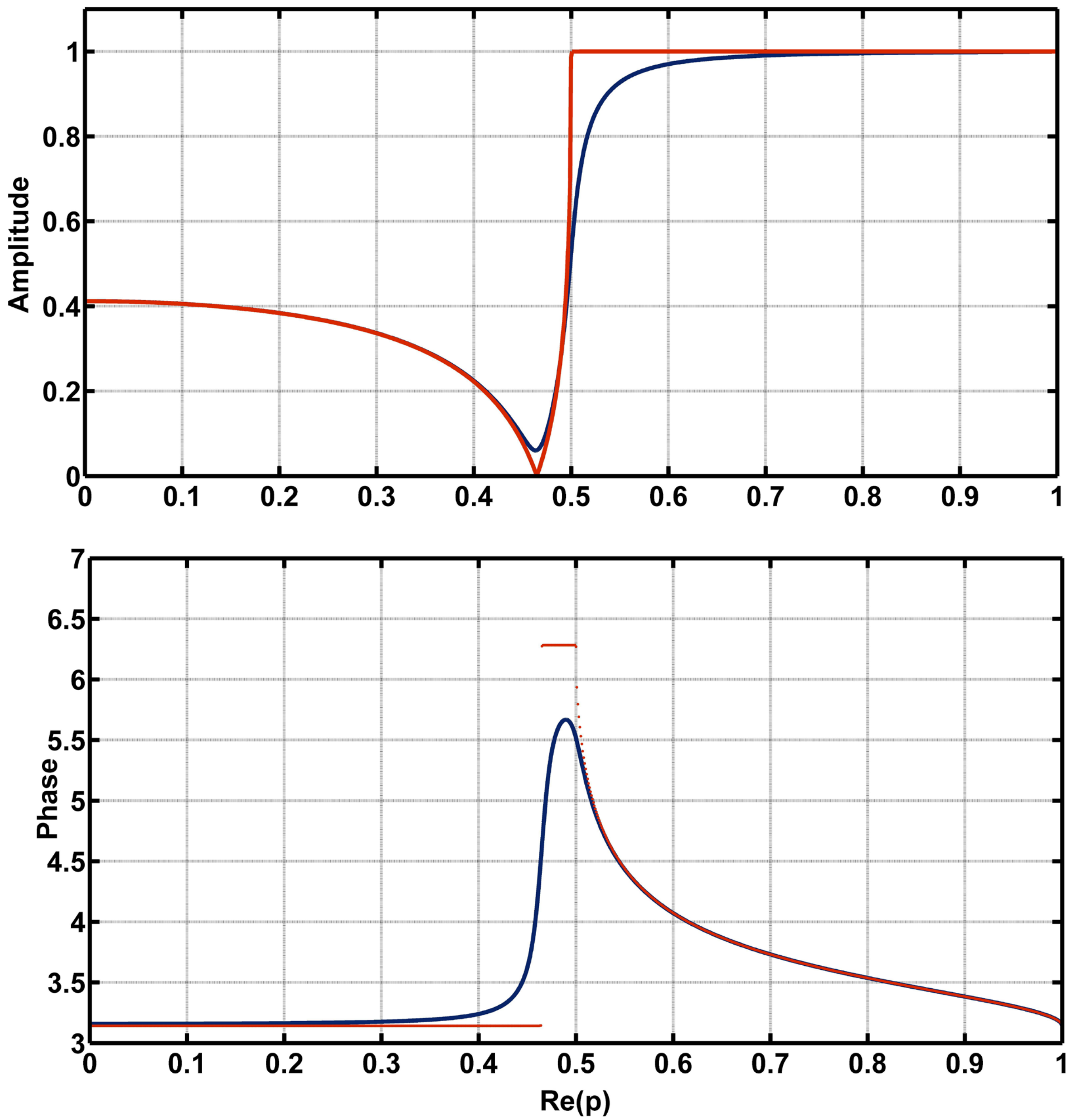

Fig. 2. The R_{11}^{SH} reflection coefficient at an interface between two medium. The upper (1) medium is elastic while the lower (2) medium is poroviscoelastic with Q_2 = 30. Medium parameters are given in Table 1.

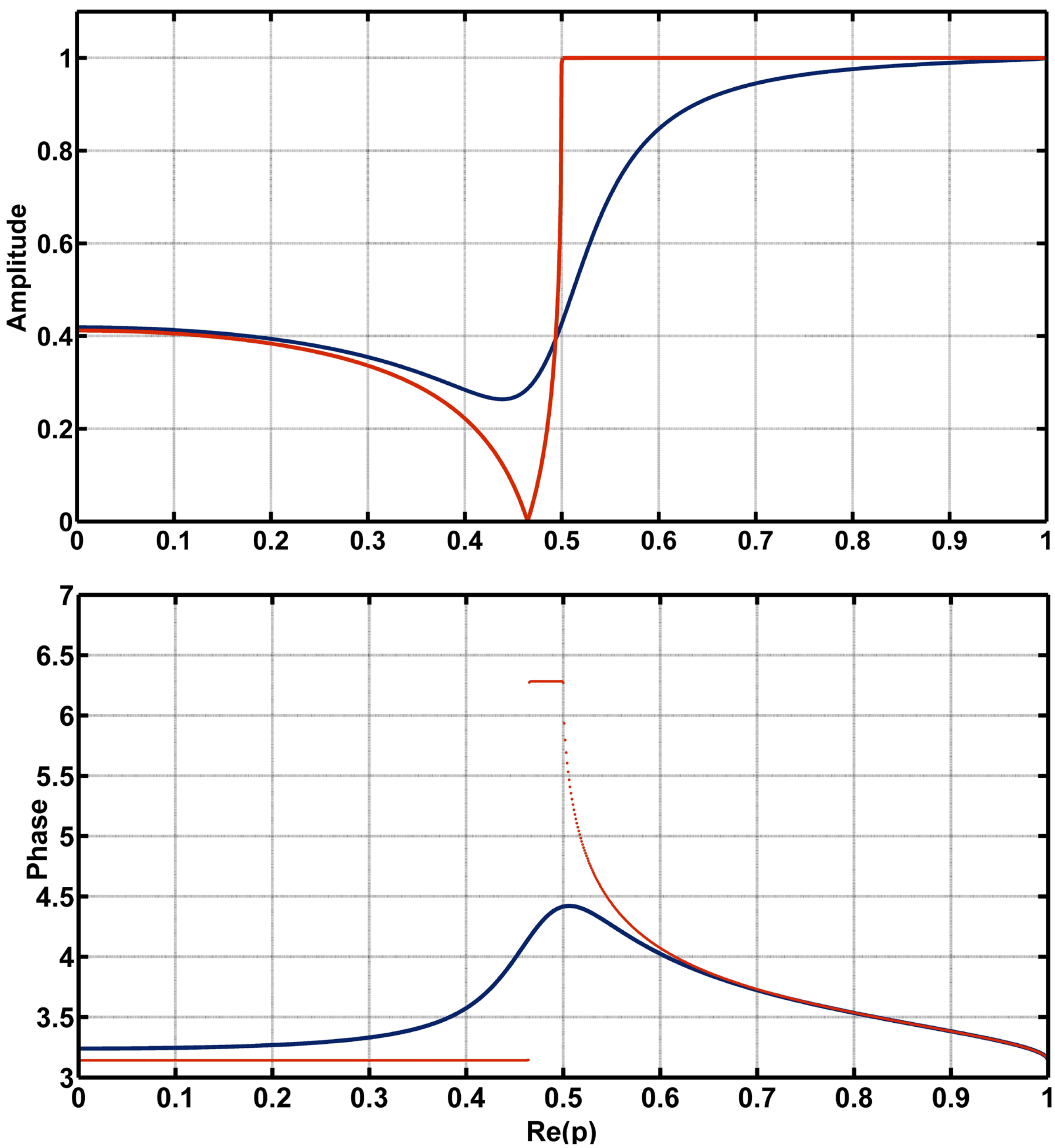

Fig. 3. The R_{11}^{SH} reflection coefficient at an interface between two medium. The upper (1) medium is elastic while the lower (2) medium is poroviscoelastic with Q_2 = 5. Medium parameters are given in Table 1.

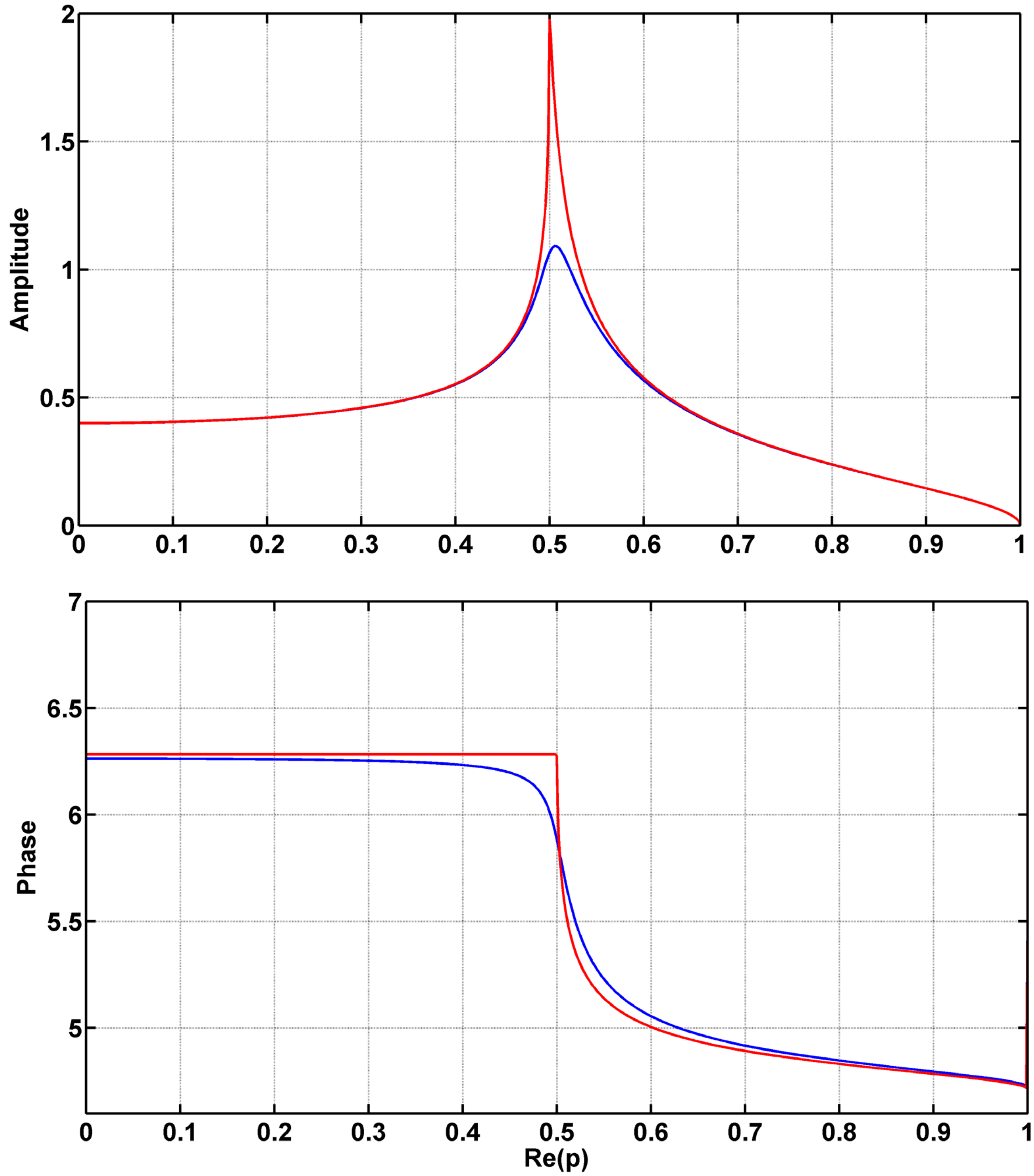

Fig. 4. The R_{12}^{SH} transmission coefficient at an interface between two medium. The upper (1) medium is elastic while the lower (2) medium is poroviscoelastic with $Q_2 = 30$. Medium parameters are given in Table 1.

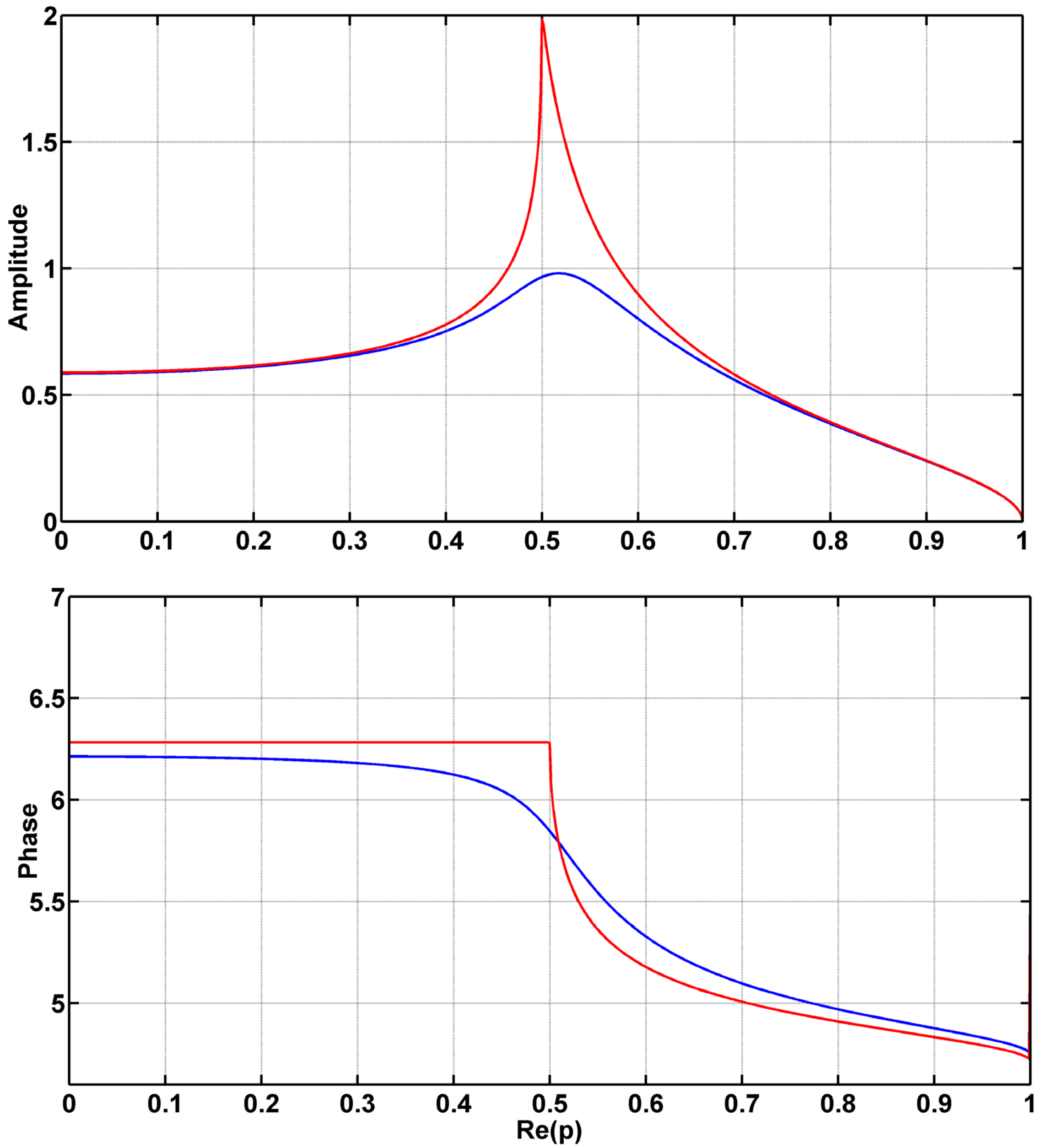

Fig. 5. The R_{12}^{SH} transmssion coefficient at an interface between two medium. The upper (1) medium is elastic while the lower (2) medium is poroviscoelastic with Q_2 = 5. Medium parameters are given in Table 1.

BRANCH POINTS AND RIEMANN SHEETS – THE DIFFICULT CASE

The *parameter* contrast at the interface between the two poroviscoelastic media for the difficult case is $\beta_2 > \beta_1$ and $Q_2 > Q_1$ where the incident/reflection (upper) medium is denoted as 1 and the medium of transmission (lower) as 2. As has been shown in many previous works on this problem, the location of the saddle point for all source receiver offsets $\left(0 \le r < \infty\right)$ lies along the straight line from the origin of the $p-$plane to the point $p_1 = \dfrac{1}{\beta_1}\left(1+\dfrac{i}{Q_1}\right)^{1/2}$.

When considering a problem whose solution is in the complex plane or some part of it, care has to be taken when the solution approaches a singularity or discontinuity. As is the case when the solution space is the real axis, discontinuities in some quantity in the solution will produce a questionable solution. The question to be asked is probably: *Is the discontinuity inherent in the quantity involved in the solution or is it reasonable to assume that the quantity is continuous over the total space of the solution?* An example of this is crossing a branch cut. From the example in the previous section, it would appear no. In the cases presented there, where the upper (incident – 1) layer was not poroviscoelastic, the value of p corresponding to all incident angles from 0 to $\pi/2$ are those real values of p lying along the real $p-$axis, $\left(0 \le p_0 \le p_1\right)$. Box 6.2 in Aki and Richards (1980) gives a brief discussion of what appears next. Other references to this may be found in almost any 3rd or 4th year mathematics texts related to the theory of complex functional analysis.

The complex valued radical q_2, defined in the complex $p-$plane, may be written as

$$q_2 = \left(p_2^2 - p^2\right)^{1/2} = \left(p_2 + p\right)^{1/2}\left(p_2 - p\right)^{1/2} \tag{15}$$

and has branch points at $p = p_2$ and $p = -p_2$. As only p values in the upper right (first) quadrant are of interest in what is considered here, the branch point at $p = p_2$ is of necessary concern. Before proceeding it should be noted that both branch points have *related* branch points at $p = \pm\infty$ in the upper and lower half planes (manifolds) of the complex $p-$plane. Any path from $p = p_2$ to some point at infinity may be taken as a branch cut. Here, a more specific requirement will be invoked: *The branch cut from* $p = p_2$ *to* $p = +\infty$ *will coincide with the path defined by the relationship* $\mathrm{Im}\left(q_2\right) = 0$*, which defines the branch cut associated with the branch point,* $p = p_2$.

It may be useful now to consider a reference function that is analytic and continuous in the first quadrant of the complex $p-$plane:

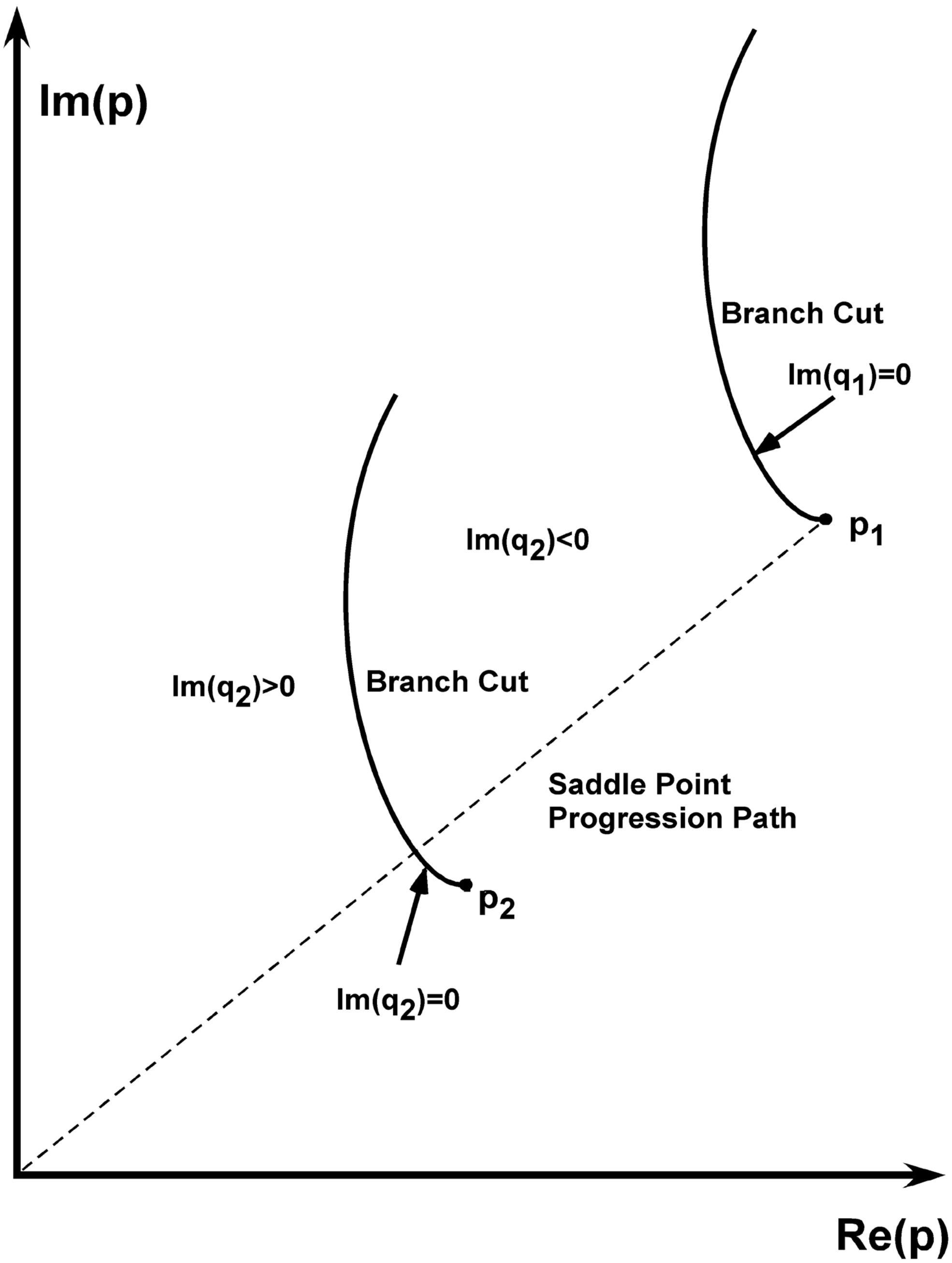

Fig. 6. This schematic shows the saddle point path for the zero – order saddle point approximation for the SH wave equation for shear wave reflection from the plane interface between the two media. The parameter values in Table 2 indicate that the saddle point path should lie along the line from the origin to p_1. This path requires that the branch cut associated with $p = p_2$ be crossed. The radical q_2 is required to remain on the first Riemann sheet as a result of arguments presented in the text

$$q_2^2 = \left(p_2^2 - p^2\right) = \left|q_2^2\right| e^{i\phi}$$
(16)

where $\left|q_2^2\right|$ is the modulus of this generally complex quantity and $\phi = \tan^{-1}\left[\frac{\mathrm{Im}\left(q_2^2\right)}{\mathrm{Re}\left(q_2^2\right)}\right]$ is its

phase. Introduce the related functions

$$\left(q_2^2\right)_m = \left|q_2^2\right| e^{i\phi + i2m\pi},$$
(17)

under the assumption that no other singularities are present in the vicinity of the existence of this function. The 2π rotations are taken in a counterclockwise sense around the branch point at $p = p_2$ and the first 3 *m* functions in the series are

$$\begin{aligned} \left(q_2^2\right)_0 &= \left|q_2^2\right| e^{i\phi} \\ \left(q_2^2\right)_1 &= \left|q_2^2\right| e^{i\phi + i2\pi}. \\ \left(q_2^2\right)_2 &= \left|q_2^2\right| e^{i\phi + i4\pi} \end{aligned}$$
(18)

With this it follows that the first 3 *m* functions associated with q_2 are

$$\begin{aligned} \left(q_2\right)_0 &= \left|q_2\right| e^{i\phi/2} \\ \left(q_2\right)_1 &= \left|q_2\right| e^{(i\phi + i2\pi)/2} = -\left|q_2\right| e^{i\phi/2} \\ \left(q_2\right)_2 &= \left|q_2\right| e^{(i\phi + i4\pi)/2} = \left|q_2\right| e^{i\phi/2} = \left(q_2\right)_0 \end{aligned}.$$
(19)

The above 3 equations are standard in complex variable analysis and possibly unnecessary to repeat here. However, they do explain the fact that q_2 may be defined on two Riemann sheets in the upper right hand quadrant of the complex $p-\text{plane}$. It may be further noticed that in the first quadrant the quantity q_2 may be forced to stay on the first sheet if the complex conjugate value of q_2 is used after crossing the branch cut, as pointed out in Krebes and Daley (2007). As q_2^2 is a continuous analytic function in the first quadrant of the complex $p-\text{plane}$, it is reasonable, based on previous arguments to require that q_2 have similar properties.

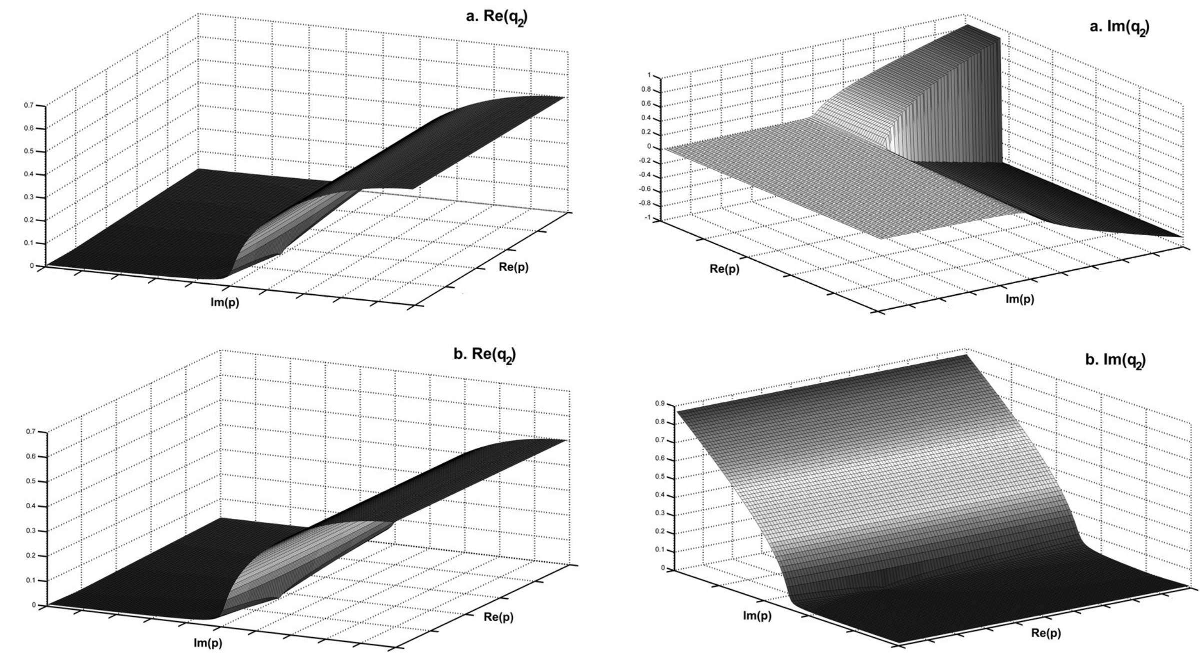

Fig. 7. The real and imaginary values of the multivalued function $q_2 = \left(p_2^2 - p^2\right)^{1/2}$ on the two possible Riemann sheets. For computational purposes sheet (b) where both real and imaginary parts of q_2 are continuous. Close examination of the individual figures reveals the position of the point p_2.

	Shear Wave Velocity (km/s)	**Density (gm/cm^3)**	**Q**
Upper (1) Medium	1.0	1.0	20
Lower (2) Medium	2.0	1.2	30/50

Table 2. Medium parameters for the difficult case, taken from Table 1. The two models here are for Q_2 = 30 and 50 with Q_1 = 20 for both.

All of this section may be written may be written more generally for the specific problem considered. Let $h(p) = \left(p_2^2 - p^2\right)$ be an analytic, continuous function defined in the first quadrant of the complex p-plane. Let the definition of a related function be given as $\left(h(p)\right)^{1/n} = \left(p_2^2 - p^2\right)^{1/n} = \left(p_2^2 - p^2\right)^{p/q}$, where n, p and q are integers and p and q have no common divisors except unity. For this problem it will further assumed that n, p and q are all positive.

Standard complex functions methods have

$$\left(h(p)\right)^{1/n} = \left|h(p)\right|^{1/n} \exp\left[\frac{i\phi}{n} + \frac{i2\pi N}{n}\right] \text{ for } N \text{ a positive integer}. \qquad (20)$$

Any source of ambiguity in the above equation, resulting in $\left(h(p)\right)^{1/n}$ being a multivalued function is not due to the amplitude, $\left|h(p)\right|^{1/n}$, but rather to the fact that there are infinitely many choices for the phase, with the principle value defined with phase $\exp\left[i\phi/n\right]$ and $\phi = \tan^{-1}\left[\mathrm{Im}\left(h(p)\right)/\mathrm{Re}\left(h(p)\right)\right]$.

What is required here is that $\left(h(p)\right)^{1/n}$ is also continuous in the first quadrant of the p-plane. Let $\left(h(p_2)\right)^{1/n}$ be the value of $\left(h(p)\right)^{1/n}$ at p_2. Construct a circle of finite size (radius - ε) centered at p_2. It will be assumed that no other singularity of $\left(h(p)\right)^{1/n}$ lies within this circle. Choose another point on this circle at say $p = \hat{p},\ \mathrm{Re}(\hat{p}) > \mathrm{Re}(p_2)$, so that the function value here is $\left(h(\hat{p})\right)^{1/n}$. Proceed along the circle in a counterclockwise direction until the point $\left(h(\hat{p})\right)^{1/n}$ is again reached. At this point $\left(h(\hat{p})\right)^{1/n}$ will return to its initial value or it will not. If one keeps this process up, circling the point p_2 a total of N times, where the value of $\left(h(\hat{p})\right)^{1/n}$ returns to its original value, it may be said that in the limit as $\varepsilon \to 0$, p_2 is a branch point of order $(N-1)$. For the case being considered here $\left(h(p)\right)^{1/2} = \left(p_2^2 - p^2\right)^{1/2},\ N = 2$. This may be observed in equation (19).

Similar to the previous section, reflection and transmission coefficients will be presented for the model described in Table 2. Again the anelastic case is shown in blue and the elastic case in red. The reflection coefficients are shown in figures (8) and (9), with the related transmission coefficients given in figures (10) and (11).

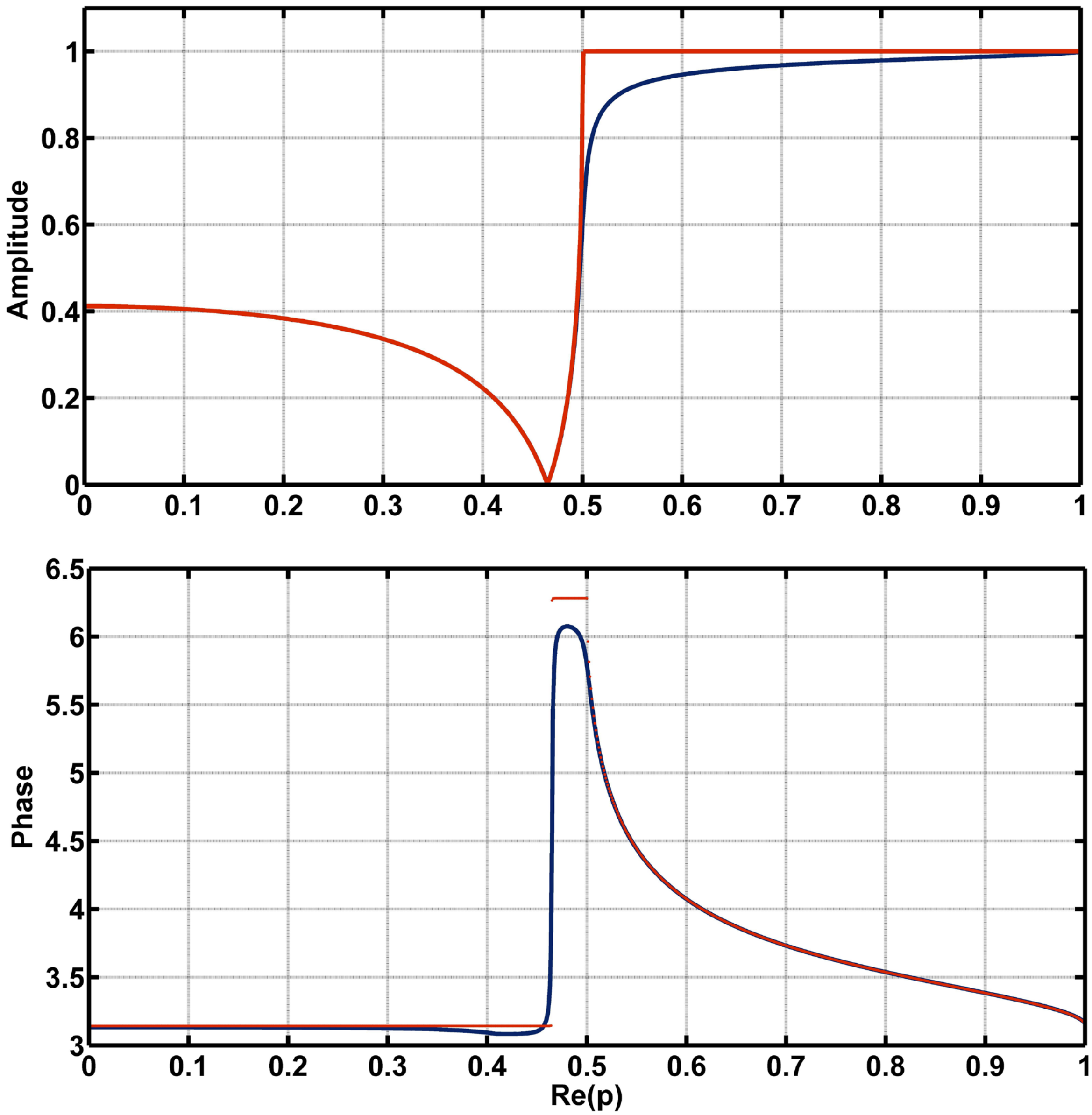

Fig. 8. The R_{11}^{SH} reflection coefficient at an interface between two medium. Both the upper (1) medium and the lower (2) media are anelastic with Q_1 = 20 and Q_2 = 50. Medium parameters are given in Table 2.

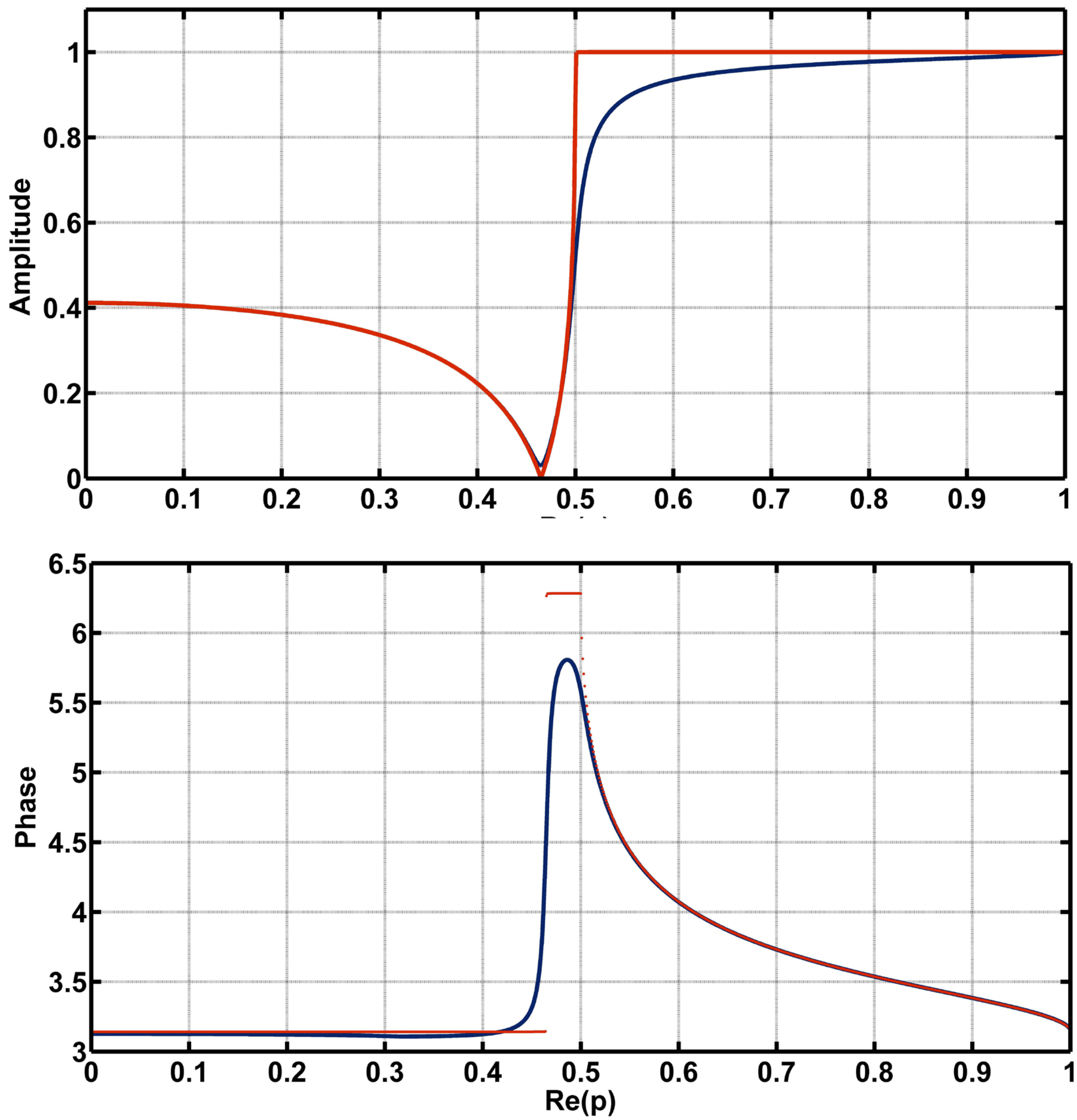

Fig. 9. The R_{11}^{SH} reflection coefficient at an interface between two medium. Both the upper (1) medium and the lower (2) media are anelastic with $Q_1 = 20$ and $Q_2 = 30$. Medium parameters are given in Table 2.

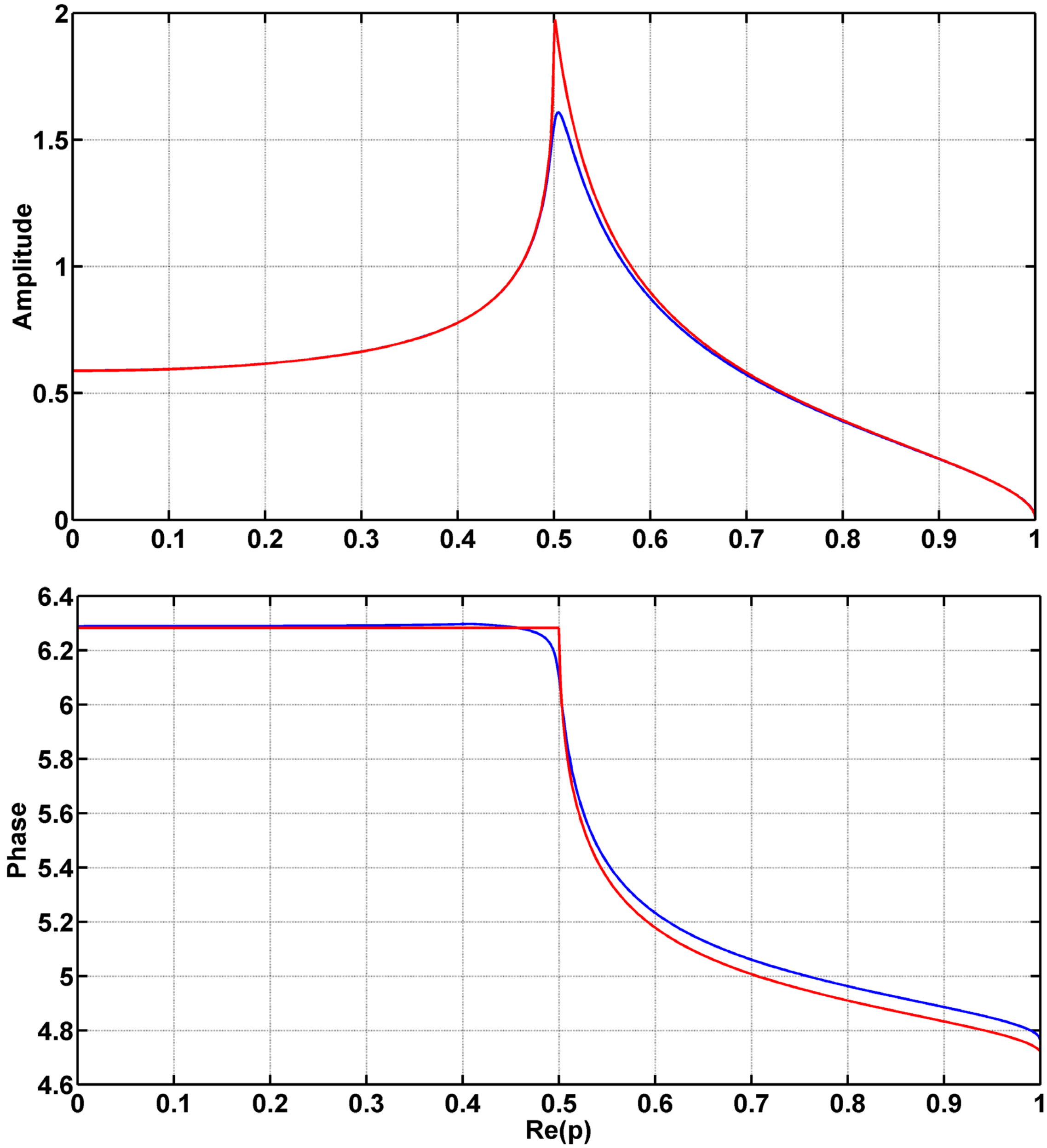

Fig. 10. The R_{12}^{SH} transmssion coefficient at an interface between two anelastic media. The upper (1) medium has Q_1 = 20 while in the lower (2) medium Q_2 = 30. Medium parameters are given in Table 2.

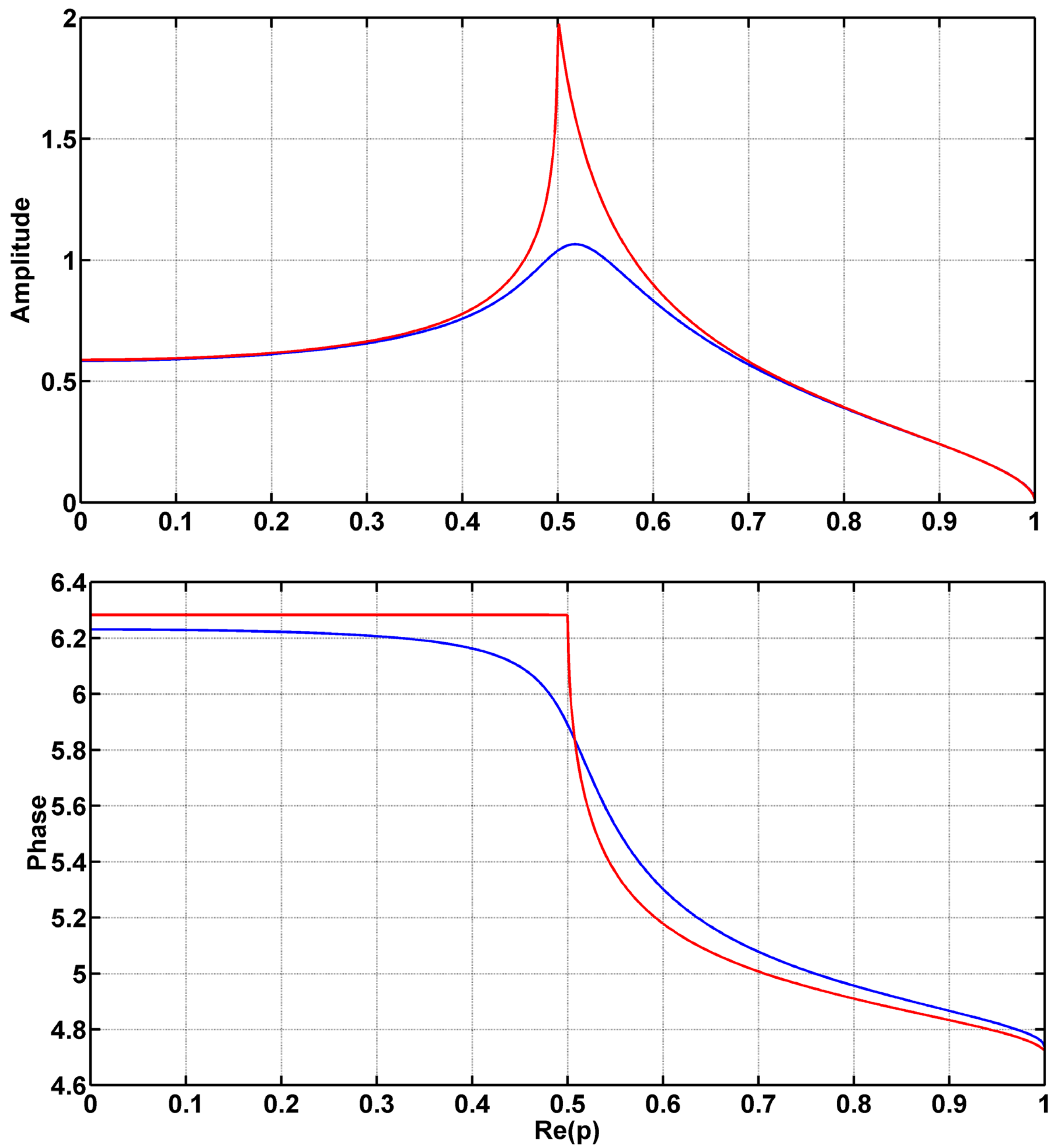

Fig. 11. The R_{12}^{SH} transmssion coefficient at an interface between two anelastic media. The upper (1) medium has Q_1 = 20 while in the lower (2) medium Q_2 = 5. Medium parameters are given in Table 2.

CONCLUSIONS

A method for dealing with wave propagation in attenuating media was investigated using equations that originated with Biot and earlier with Frenkel. Using the theory presented by these authors the often used concept of complex velocities is eliminated, and the attenuation is achieved through the introduction of term specific to that physical phenomena. The writer of this has for many decades had difficulties with complex velocity, what its physical meaning was and how it was incorporated into theories based on equations defining wave propagation in an elastic media. There are hundreds, if not thousands, of papers in the literature based on the hypothesis of complex velocities. This report was not written to question the validity of these previous works, but rather to introduce an alternative that might be of use in specific situations.

REFERENCES

Aki, K. & Richards, P.G., 1980. Quantitative Seismology, Freeman, San Francisco, CA.

Aki, K. & Richards, P.G., 2002. Quantitative Seismology, 2nd edition, University Science Books, Sausalito, CA.

Carcione, J.M., 2007. *Wave Fields in Real Media: Wave Propagation in Anisotropic Anelastic, Porous, and Electromagnetic Media*, 2nd edition Elsevier.

Hildebrand, F.B., 1962, Advanced calculus for applications, Prentice – Hall, New Jersey.

Krebes, E.S. and Daley, P.F., 2007. Difficulties with computing anelastic plane wave reflection and transmission coefficients, *Geophys. J. Int.*, **170**, 205–216.

Morozov, I.B., 2009a. Thirty years of confusion around "scattering *Q*"? *Seismological Research Letters*, 80, 5–7.

Morozov, I.B., 2009b. More reflections on *Q*. *CSEG Recorder*, **34**, 12–13.

Morozov, I.B., 2010. Exact elastic *P*/*SV* impedance. *Geophysics*, **75**, C7–C13.

Morozov, I.B. , 2011. Anelastic acoustic impedance and the correspondence principle, Geophysical *Prospecting,* **59**, 24–34

Morozov, I.B. and Ma, J., 2009. Accurate post-stack seismic inversion by well-log calibration. *Geophysics*, 74, R59–R67.

Lines, L., Vasheghani, F. and Treitel, S., 2008. Reflections on *Q*. *CSEG Recorder*, **33**, 36–38.

Ruud, B.O., 2006. Ambiguous reflection coefficients for anelastic media, *Studia Geophysica et Geodætica*, **50**, 479–498.