Viscoelastic scattering potentials and inversion sensitivities

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ABSTRACT

The scattering formulation presented in 2012 by Stolt and Weglein for isotropic elastic waves, in addition to being a major contribution on its own, represents a jumping-off point for the description of the interaction of seismic waves with a wide range of heterogeneous media. In this paper we extend the scattering picture to include viscoelastic waves, focusing in particular on generalizing the layered medium results of Borcherdt. The main theoretical challenge involves the choice of coordinate system over which to evaluate and analyze the waves, which in the viscoelastic case must be based on complex vector analysis. However, a good candidate system is determined, within which several of Borcherdt's key results for layered media, concerning the reflection and conversion of homogeneous and inhomogeneous P- and S-wave modes, are shown to carry over to general multidimensional scattering. In addition to extending the domain of applicability of this type of viscoelastic wave theory in an intuitive way, the results when incorporated in a Born approximate wave data model are immediately applicable to both direct and full-waveform type inversion calculations.

INTRODUCTION

Stolt and Weglein (2012) have introduced a formal theory for the description of the multidimensional scattering of seismic waves based on an isotropic-elastic model. We identify as a research priority the adaptation of this approach to incorporate other, more complete pictures of seismic wave propagation. Amongst these, the extension to include anelasticity and/or viscoelasticity (Flugge, 1967), which brings to the wave model the capacity to transform elastic energy into heat, ranks very high. Anelasticity is generally held to be a key contributor to seismic attenuation, or "seismic Q", which has received several decades worth of careful attention in the literature (e.g., Aki and Richards, 2002; Futterman, 1967). Development of methods for analysis (e.g., Tonn, 1991), processing (Bickel and Natarajan, 1985; Hargreaves and Calvert, 1991; Wang, 2006; Zhang and Ulrych, 2007; Innanen and Lira, 2010), and inversion (Dahl and Ursin, 1992; Ribodetti and Virieux, 1998; Causse et al., 1999; Hicks and Pratt, 2001; Innanen and Weglein, 2007) of wave data exhibiting the attenuation and dispersion of seismic Q remains a very active research area.

Borcherdt (2009) has presented a complete theory for seismic waves propagating in layered anelastic media, assuming a viscoelastic model to hold. Borcherdt's formulation is particularly powerful in that it predicts a range of transverse, inhomogeneous wave types unique to viscoelastic media (Type I and II S waves), and develops rules for conversion of one type to another during interactions with planar boundaries.

In the elastic-isotropic setting, beginning with a plane defined by the incoming wave vector and the outgoing wave vector, Stolt and Weglein (2012) were able to develop scattering quantities which in an intuitive manner generalized the layered-medium notions of reflections and conversions of P, Sv and Sh waves. The results are forms for the scattering operator whose diagonal elements describe the potential of a volume scattering element to scatter a P wave to a P wave, an Sv to an Sv wave, and an Sh to an Sh wave, and whose off-diagonal elements describe the potential to convert, from, say, a P wave to an Sv wave, etc. Having made a "good" choice of coordinate systems, canonical results, such as the lack of P-Sh and Sv-Sh mode conversions, are naturally reproduced in their formulation: the off-diagonal element corresponding to P-Sh scattering is seen to be identically zero.

Generalizing this approach to allow for viscoelastic waves of the type described by Borcherdt has several positive outcomes. First, and foremost, it provides an analytical framework for the examination of processes of scattering of viscoelastic waves from arbitrary three-dimensional heterogeneities, as opposed to layered media. Second, it provides a foundation for direct linear and nonlinear inversion methods for reflection seismic data, which go well beyond existing an-acoustic results (Innanen and Weglein, 2007; Innanen and Lira, 2010). And third, it provides a means to compute and analyze the gradient and Hessian quantities used in iterative seismic inversion (see the review by Virieux and Operto, 2009).

The main challenge of the viscoelastic generalization of multidimensional scattering lies in the need to choose from a much wider range of possible systems of coordinates. Because the viscoelastic wave vectors are complex, and the real and imaginary components of these wave vectors are not necessarily parallel, the use of incident and scattered wave vectors as the starting point for coordinate system selection is a richer but more complicated idea. Nevertheless, judicious choices are possible, and we arrive at a complex, or bivector coordinate system which appears to naturally extend the concepts of Borcherdt (2009) to arbitrary 3D scattering.

THE MATHEMATICS OF HOMOGENEOUS AND INHOMOGENEOUS VISCOELASTIC WAVES

Inhomogeneous waves play an important role in viscoelastic wave theory. Such waves, which decrease in amplitude with propagation distance, are described with the use of complex vectors or *bivectors* (Morro, 1992). They are expressed as the product of a bivector and a exponential factor. The real part of this product is interpreted as the displacement vector. The bivector analysis which is used to arrive at results in viscoelastic wave theory differs from real vector analysis like complex analysis differs from real analysis. We will discuss in as intuitive a way as possible these differences as they arise.

There are three types of waves in a viscoelastic medium: P, Type-I S, and Type-II S. For each wave type there is a corresponding seismic quality factor, Q_P , Q_{SI}, and Q_{SII}. In the special case of homogenous waves, Q_P , Q_{SI}, and Q_{SII} reduce to Q_{HP} and Q_{HS} (Borcherdt, 2009). These quality factors have the standard definitions in terms of ratios of the real and imaginary parts of the complex moduli.

In this paper we will write quantities such as the viscoelastic wave vector and velocity for inhomogeneous waves in terms of the reciprocal quality factors for *homogenous waves*, i.e., Q_{HP} and Q_{HS}. Of the several mathematical possibilities this choice seems to be the most convenient, expressing our results in the "language" of standard exploration and monitoring seismology. As a consequence, the key result of this paper, the enumeration of the explicit elements of the multidimensional viscoelastic scattering operator, appears in terms of perturbations in Q_{HP} and Q_{HS}. These perturbations correspond to the relative-change quantities involved in anelastic amplitude-variation-with-angle (AVA) and amplitude-variation-with-frequency (AVF) expressions.

In the case of inhomogeneous waves, the attenuation and propagation vectors are not in the same direction. This makes the displacement vectors different from homogenous case. In what follows, we show that the particle motion for P waves is elliptical in the plane constructed by attenuation and propagation vectors. This elliptical motion reduces to a linear motion in the limit of homogenous case. Also, we have the two types of shear waves SI and SII. The first one, which is the generalization of SV wave, has an elliptical displacement vector in the propagation-attenuation plane. Finally SII type wave which is a generalization of SH wave types has a linear motion perpendicular to the propagation-attenuation plane. The wavenumber vector of inhomogeneous waves is represented by

$$\mathbf{K} = \mathbf{P} - i\mathbf{A}. \tag{1}$$

Here $\mathbf{P}$ is the propagation vector perpendicular to the constant phase plane $\mathbf{P}\cdot\mathbf{r} = constant$, and $\mathbf{A}$ is the attenuation vector perpendicular to the amplitude constant plane $\mathbf{A}\cdot\mathbf{r} = constant$. In the case that attenuation and propagation vectors are in the same direction, wave is homogeneous. Elastic media is represented by $\mathbf{A} = 0$. If we represent the angle between $\mathbf{P}$ and $\mathbf{A}$ by δ, for inhomogeneous waves $\delta \neq 0$ we have

$$|\mathbf{P}| = 2^{-\frac{1}{2}}\left[\Re\mathbf{K}\cdot\mathbf{K} + \sqrt{(\Re\mathbf{K}\cdot\mathbf{K})^2 + (\Im\mathbf{K}\cdot\mathbf{K})^2\sec^2\delta}\right]^{\frac{1}{2}}, \tag{2}$$

and

$$|\mathbf{A}| = 2^{-\frac{1}{2}}\left[-\Re\mathbf{K}\cdot\mathbf{K} + \sqrt{(\Re\mathbf{K}\cdot\mathbf{K})^2 + (\Im\mathbf{K}\cdot\mathbf{K})^2\sec^2\delta}\right]^{\frac{1}{2}} \tag{3}$$

where

$$\mathbf{K}\cdot\mathbf{K} = |\mathbf{P}|^2 - |\mathbf{A}|^2 - 2i|\mathbf{P}||\mathbf{A}|\cos\delta = \left(\frac{\omega}{\alpha}\right)^2 = \frac{\omega^2\rho}{K + \frac{4}{3}M}. \tag{4}$$

Here K and M are the viscoelastic Lamé parameters. From this general framework we may now follow Borcherdt (2009) in analyzing three types of independently propagating wave.

Viscoelastic P waves

We wish to characterize the particle motions of the various viscoelastic waves. First we consider the P-wave. In this case the displacement vector is given by

$$\mathbf{u}_P = \Re\left\{-i\mathbf{K}_P U_P e^{i(\omega t - \mathbf{K}_P\cdot\mathbf{r})}\right\}. \tag{5}$$

To obtain the particle motion we define the two vectors $\mathbf{b}_P$ and $\mathbf{c}_P$ as

$$\mathbf{b}_P = \frac{K_P^R\mathbf{P}_P - K_P^I\mathbf{A}_P}{{K_P^R}^2 + {K_P^I}^2}, \tag{6}$$

and

$$\mathbf{c}_P = \frac{K_P^I \mathbf{P}_P - K_P^R \mathbf{A}_P}{{K_P^R}^2 + {K_P^I}^2}, \tag{7}$$

where

$$K_P = \sqrt{\mathbf{K}_P \cdot \mathbf{K}_P} = K_P^R + iK_P^I. \tag{8}$$

The displacement vector is then

$$\mathbf{u}_P = B_P \left(\mathbf{b}_P \cos \Omega_P(t) + \mathbf{c}_P \sin \Omega_P(t) \right), \tag{9}$$

where

$$B_P = |U_P K_P| e^{-\mathbf{A}_P \cdot \mathbf{r}}, \tag{10}$$

and

$$\Omega_P(t) = \omega t - \mathbf{P}_P \cdot \mathbf{r} + \arg\left(U_p K_P\right) - \pi/2. \tag{11}$$

By the orthogonality of $\mathbf{b}_P$ and $\mathbf{c}_P$ we have

$$\frac{u_{c_P}^2}{[c_P \exp(-\mathbf{A}_P \cdot \mathbf{r})]^2} + \frac{u_{b_P}^2}{[b_P \exp(-\mathbf{A}_P \cdot \mathbf{r})]^2} = 1, \tag{12}$$

where u_{c_P} and u_{b_P} are the components of $\mathbf{u}_P$ in the direction of $\mathbf{b}_P$ and $\mathbf{c}_P$. From equation (5) we find that the particle motion is elliptical in the plane of ($\mathbf{P}_P - \mathbf{A}_P$), with the major axis $B_P \mathbf{b}_P$, minor axis $B_P \mathbf{c}_P$, and eccentricity b_P^{-1}. We can also write the displacement vector in terms of propagation and attenuation vectors (Borcherdt, 2009):

$$\mathbf{u}_P = B_P \left(\frac{\mathbf{P}_P}{|K_P|} \cos[\Omega_P(t) + \Lambda_P] + \frac{\mathbf{A}_P}{|K_P|} \sin[\Omega_P(t) + \Lambda_P] \right) \tag{13}$$

where

$$\Lambda_P = \tan^{-1} \left[\frac{Q_{HP}^{-1}}{1 + \sqrt{1 + Q_{HP}^{-2}}} \right]. \tag{14}$$

For low-loss viscoelastic media where $Q_{HP}^{-1} \ll 1$, Λ_P reduces to $Q_{HP}^{-1}/2$.

Viscoelastic SI waves

In the case of S-waves, Borcherdt distinguishes types I and II waves by assuming forms for the complex displacement amplitude vector in the general wave solutions. For S-waves the displacement vector is given by

$$\mathbf{u}_S = \Re \left\{ -i\mathbf{K}_S \times \mathbf{U}_S e^{i(\omega t - \mathbf{K}_S \cdot \mathbf{r})} \right\}, \tag{15}$$

in which $\mathbf{U}_S$ is a general complex vector

$$\mathbf{U}_S = U_{Sx}\mathbf{x} + U_{Sy}\mathbf{y} + U_{Sz}\mathbf{z}. \tag{16}$$

If we assume that it has the form $\mathbf{U}_S = U_S \mathbf{n}$, where $\mathbf{n}$ is a unit vector orthogonal to the plane of $\mathbf{P}_S - \mathbf{A}_S$, the corresponding wave is named SI-type wave. It should be noted that this orthogonality is in the sense of complex vector analysis, and does not imply that the displacement vector is perpendicular to the propagation wave number vector $\mathbf{P}_S$ (see Appendix A). We can write the displacement vector for SI-type wave as follows:

$$\mathbf{u}_{SI} = B_S \left[\mathbf{b}_S \cos\Omega_S(t) + \mathbf{c}_S \sin\Omega_S(t)\right], \tag{17}$$

where the two vectors $\mathbf{b}_S$ and $\mathbf{c}_S$ are defined as

$$\mathbf{b}_S = \frac{K_S^R \mathbf{P}_S - K_S^I \mathbf{A}_S}{K_S^{R^2} + K_S^{I^2}} \times \mathbf{n}, \tag{18}$$

and

$$\mathbf{c}_S = \frac{K_S^I \mathbf{P}_S - K_S^R \mathbf{A}_S}{K_S^{R^2} + K_S^{I^2}} \times \mathbf{n}, \tag{19}$$

and

$$B_S = |\mathbf{U}_S K_S| e^{-\mathbf{A}_S \cdot \mathbf{r}}, \tag{20}$$

$$\Omega_S(t) = \omega t - \mathbf{P}_S \cdot \mathbf{r} + \arg(U_S K_S) + \pi/2. \tag{21}$$

We additionally define

$$\mathbf{n} = \frac{\mathbf{P}_S \times \mathbf{A}_S}{|\mathbf{P}_S \times \mathbf{A}_S|}, \tag{22}$$

$$K_S = \sqrt{\mathbf{K}_S \cdot \mathbf{K}_S} = K_S^R + iK_S^I. \tag{23}$$

SI waves, like P-waves, hence involve elliptical particle motion. In the special case of a homogeneous wave this elliptical motion reduces to linear motion perpendicular to the wave propagation direction. We can also write the displacement vector in terms of propagation and attenuation vectors

$$\mathbf{u}_S = B_S \left(\frac{\mathbf{P}_S \times \mathbf{n}}{|K_S|} \cos[\Omega_S(t) + \Lambda_S] + \frac{\mathbf{A}_S \times \mathbf{n}}{|K_S|} \sin[\Omega_S(t) + \Lambda_S] \right), \tag{24}$$

where

$$\Lambda_S = \tan^{-1}\left[\frac{Q_{HS}^{-1}}{1 + \sqrt{1 + Q_{HS}^{-2}}}\right], \tag{25}$$

which for low-loss viscoelastic media reduces to $Q_{HS}^{-1}/2$.

Viscoelastic SII waves

Next, we assume that $\mathbf{U}_S$ is not simply a complex number multiply by a real unit vector but has a general form as (16). In this case, the corresponding wave is defined as SII-wave and sometimes linear S-waves. As we will show, the particle motion for SII-wave is linear for both homogeneous and inhomogeneous waves and perpendicular to the wavenumber vector $\mathbf{K}_S$. Without loss of generality assume that $\mathbf{U}_S$ is in the xz plane, accordingly the displacement vector takes the form

$$\mathbf{u}_{SII} = \Re\left\{ i\mathbf{K}_S \cdot (U_{Sz}\mathbf{x} - U_{Sx}\mathbf{z}) e^{i(\omega t - \mathbf{K}_S \cdot \mathbf{r})} \right\} \mathbf{y}, \tag{26}$$

equation (26) indicates that the particle motion for SII wave is linear perpendicular to the $(\mathbf{P}_S - \mathbf{A}_S)$-plane.

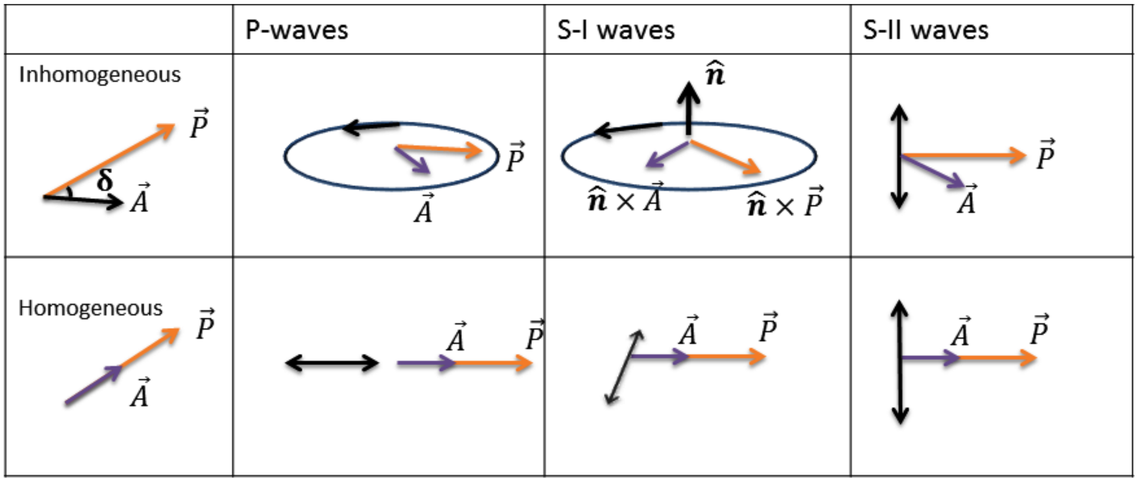

FIG. 1. Diagram illustrating the particle motion orbit of waves in a viscoelastic media. For P-waves the particles moves on an ellipse, in the homogeneous case, then the elliptical motion reduces to the linear motion in the direction of propagation vector $\mathbf{P}$. Similar to P-waves, for SI waves the motion of particle is elliptical and for the homogeneous wave the elliptical particle motion collapse to a linear motion perpendicular to the direction of propagation vector $\mathbf{P}$. For SII waves in both case of homogeneous and inhomogeneous waves the particle motion is linear perpendicular to the direction of propagation vector $\mathbf{P}$.

THE VISCOELASTIC SCATTERING OPERATOR AND POTENTIALS

Scattering theory is a framework within which various kinds of interactions of waves and particles can be analyzed. In the context of seismic exploration, scattering theory studies how perturbations in the properties of the medium are related to the seismic waves that propagate through those perturbations (e.g., Weglein et al., 2003). The perturbations are assembled, with reference medium properties, in a core quantity call the scattering operator, the construction of which for viscoelastic waves will be the subject of this section.

The seismic scattering formulation, and the resulting scattering operator forms, can be used to generalize "layered medium" wave propagation results, providing expressions describing waves interacting with not 1D media but with arbitrary multidimensional heterogeneities. It can be used in principle to generate exact solutions for such waves, but those solutions are in the form of infinite series, which are subject to often rather thorny questions of convergency. In fact the main application has been in the generation of powerful approximate solutions.

The Born approximation is a solution accurate to first order in the scattering operator, and is used as the basis for many types of migration and linearized inversion in seismic applications (Bleistein, 1979; Clayton and Stolt, 1981; Beylkin, 1985, etc.). Mapping between the scattering operator and the Born approximate model of seismic data usually involves integrating over all space the product of the operator with relatively simple Green's functions: the scattering operator contains most of the interesting physics. The study of the scattering operator in isolation, in other words, provides direct insight into the physics of wave interactions. It is akin to arriving at conclusions about waves in layered media by studying the mathematical structure of the P-P reflection coefficient (as in, e.g., Aki and Richards, 2002). In this section we arrive at interpretable forms of the viscoelastic scattering operator, including explicit expressions for the elements of the operator. Each element will represent the potential of a point in space to scatter a P to a P wave, a P to an SI wave, etc. Then we will be in a position to analyze the viscoelastic scattering problem for general insights.

The scattering operator in displacement space

In the scattering framework, the unperturbed medium is defined as a reference medium and the perturbed medium as a actual medium. The difference between the actual and reference medium wave operators is the perturbation operator or scattering operator. In the elastic-isotropic case, this operator is given by a 3×3 matrix, each element of which corresponds to the scattering of one wave type to another. The diagonal elements refer to scattering which conserves wave type, and off-diagonal elements refer to the those which converts wave type. To begin the process of formulating a viscoelastic framework, we express the viscoelastic wave equation as

$$L_{ve}(\mathbf{r},\omega)\mathbf{u}(\mathbf{r},\omega) = 0. \tag{27}$$

Here the wave operator in Cartesian coordinates is

$$L_{ve} = \begin{pmatrix} L_{xx} & L_{xy} & L_{xz} \\ L_{yx} & L_{yy} & L_{yz} \\ L_{zx} & L_{zy} & L_{zz} \end{pmatrix}, \tag{28}$$

with the elements

$$(L_{ve})_{ij} = \rho\omega^2\delta_{ij} + \partial_i(\rho\alpha^2)\partial_j + \delta_{ij}\partial_k(\rho\beta^2)\partial_k - 2\partial_i(\rho\beta^2)\partial_j + \partial_j(\rho\beta^2)\partial_i, \tag{29}$$

for $i, j, k = x, y, z$. The P- and S-wave velocities are defined as

$$\alpha = \sqrt{\rho^{-1}\left(K + \frac{4M}{3}\right)}, \tag{30}$$

and

$$\beta = \sqrt{\rho^{-1}M}, \tag{31}$$

where ρ is the mass density, and M and K are viscoelastic moduli which are generally complex and frequency dependent. The scattering matrix is the difference between perturbed and unperturbed wave operators of the type in equation (28):

$$V_{ve}(\mathbf{r},\omega) = L_{ve}(\mathbf{r},\omega) - L_{ve_0}(\mathbf{r},\omega) = \begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix}. \tag{32}$$

The scattering operator can be written as follows:

$$V_{ve}(\mathbf{r},\omega) = \sum_k C_k^L(\mathbf{r},\omega)A_k(\mathbf{r})C_k^R(\mathbf{r},\omega), \tag{33}$$

where $A_i(\mathbf{r})$ are the perturbation parameters, and $C_i^L(\mathbf{r},\omega)$ and $C_i^R(\mathbf{r},\omega)$ are functions of first-order derivatives. For low-loss viscoelastic media in which $Q_{HP}^{-1} \ll 1$ and $Q_{HP}^{-1} \ll 1$ we have

$$\alpha^2 = \alpha_H^2(1 + iQ_{HP}^{-1}), \tag{34}$$

and

$$\beta^2 = \beta_H^2(1 + iQ_{HS}^{-1}). \tag{35}$$

We define the perturbation parameters

$$A_\rho \approx 1 - \frac{\rho_0}{\rho} = \frac{\Delta\rho}{\rho} \simeq \frac{\Delta\rho}{\rho_0}, \tag{36}$$

$$A_{\alpha_H} = 1 - \frac{\alpha_{H_0}}{\alpha_H} = \frac{\Delta\alpha_H}{\alpha_H} \simeq \frac{\Delta\alpha_H}{\alpha_{H_0}}, \tag{37}$$

$$A_{\beta_H} = 1 - \frac{\beta_{H_0}}{\beta_H} = \frac{\Delta\beta_H}{\beta_H} \simeq \frac{\Delta\beta_H}{\beta_{H_0}}, \tag{38}$$

$$A_{Q_{HP}} = 1 - \frac{Q_{HP_0}}{Q_{HP}} = \frac{\Delta Q_{HP}}{Q_{HP}} \simeq \frac{\Delta Q_{HP}}{Q_{HP_0}}, \tag{39}$$

and

$$A_{Q_{HS}} = 1 - \frac{Q_{HS_0}}{Q_{HS}} = \frac{\Delta Q_{HS}}{Q_{HS}} \simeq \frac{\Delta Q_{HS}}{Q_{HS_0}}. \tag{40}$$

Furthermore using equations (37) and (38), we can write

$$\left(\frac{\alpha_H}{\alpha_{H_0}}\right)^2 = (1 - A_{\alpha_H})^{-2} \simeq 1 + 2A_{\alpha_H} \tag{41}$$

and

$$\left(\frac{\beta_H}{\beta_{H_0}}\right)^2 = (1 - A_{\beta_H})^{-2} \simeq 1 + 2A_{\beta_H}. \tag{42}$$

The elements of scattering potential, in terms of these perturbations, are

$$\rho_0^{-1}(V_{ve}^{\rho})_{ij} = A_\rho\omega^2\delta_{ij} + \alpha_{H_0}^2\partial_i A_\rho \partial_j + \beta_{H_0}^2\left(\delta_{ij}\partial_k A_\rho \partial_k - 2\partial_i A_\rho \partial_j + \partial_j A_\rho \partial_i\right), \tag{43}$$

$$\rho_0^{-1}(V_{ve}^{\alpha_H})_{ij} = 2\alpha_{H_0}^2\partial_i A_{\alpha_H}\partial_j, \tag{44}$$

$$\rho_0^{-1}(V_{ve}^{\beta_H})_{ij} = 2\beta_{H_0}^2\left(\delta_{ij}\partial_k A_{\beta_H}\partial_k - 2\partial_i A_{\beta_H}\partial_j + \partial_j A_{\beta_H}\partial_i\right), \tag{45}$$

$$\rho_0^{-1}(V_{ve}^{Q_{HP}})_{ij} = i\alpha_{H_0}^2 Q_{HP_0}^{-1}\partial_i A_{V_{QP}}\partial_j, \tag{46}$$

and

$$\rho_0^{-1}(V_{ve}^{Q_{HS}})_{ij} = i\beta_{H_0}^2\left(\delta_{ij}\partial_k A_{Q_{HS}}\partial_k - 2\partial_i A_{Q_{HS}}\partial_j + \partial_j A_{Q_{HS}}\partial_i\right), \tag{47}$$

where we have defined

$$V_{ve} = V_{ve}^{\rho} + V_{ve}^{\alpha_H} + V_{ve}^{\beta_H} + V_{ve}^{Q_{HP}} + V_{ve}^{Q_{HS}}. \tag{48}$$

The scattering operator in P, SI and SII space

The next task is to evaluate the scattering matrix in a system which naturally describes Borcherdt's viscoelastic modes $\mathrm{P}, \mathrm{SI}, \mathrm{SII}$, namely

$$\mathcal{V}_{ve} = \begin{pmatrix} {}^{P}_{P}\mathcal{V}_{ve} & {}^{P}_{SII}\mathcal{V}_{ve} & {}^{P}_{SI}\mathcal{V}_{ve} \\ \\ {}^{SII}_{P}\mathcal{V}_{ve} & {}^{SII}_{SII}\mathcal{V}_{ve} & {}^{SII}_{SI}\mathcal{V}_{ve} \\ \\ {}^{SI}_{P}\mathcal{V}_{ve} & {}^{SI}_{SII}\mathcal{V}_{ve} & {}^{SI}_{SI}\mathcal{V}_{ve} \end{pmatrix}. \tag{49}$$

Here the diagonal elements represent the scattering that preserves the wave types and off-diagonal elements refer to the scattering that converts the type of waves. For example ${}^{P}_{SI}\mathcal{V}_{ve}$ refer to the scattering that a P-wave converts into the SI-type wave. As we show later some elements are zero, for instance ${}^{P}_{SII}\mathcal{V}_{ve} = 0$. It means that a P-wave with elliptical polarization cannot convert into a SII wave with a linear polarization.

Now let us define a framework to transform the scattering matrix in cartesian coordinate (32) to (49). Since the particle motion for SII-type wave is in $\mathbf{n}$ direction we can define the normal polarization vector as $\mathbf{e}_{SII} = \mathbf{n}$. For SII wave, since the displacement vector moves on an ellipse in the the plane of attenuation-propagation, we define a unit complex vector $\mathbf{e}_{SI}$ as

$$\mathbf{e}_{SI} = \frac{\mathbf{K}_S \times \mathbf{n}}{|\mathbf{K}_S \times \mathbf{n}|} = \frac{\mathbf{K}_S}{|\mathbf{K}_S|} \times \mathbf{n} = \hat{\mathbf{K}}_S \times \mathbf{n}. \tag{50}$$

It can be shown that these vectors have the following orthogonality properties

$$\begin{aligned} \hat{\mathbf{K}}_S \times \mathbf{e}_{SII} &= \mathbf{e}_{SI} \\ \mathbf{e}_{SI} \times \hat{\mathbf{K}}_S &= \mathbf{e}_{SII} \\ \mathbf{e}_{SII} \times \mathbf{e}_{SI} &= \hat{\mathbf{K}}_S \\ \mathbf{e}_{SII} \cdot \mathbf{K} &= 0 \\ \mathbf{e}_{SII} \cdot \mathbf{e}_{SI} &= 0 \\ \mathbf{e}_{SI} \cdot \mathbf{K} &= 0. \end{aligned} \tag{51}$$

In order to determine the scattering matrix in terms of P, SII, and SI components we apply the method used by Stolt and Weglein (2012). In the first step by using a transformation, the matrix in the x, y and z coordinates reduces to one whose elements correspond to P-waves and the x, y and z-components of S-waves. The corresponding operators for this transformation are given by

$$\Pi_r = i \begin{pmatrix} K_{P_{rx}} & K_{P_{ry}} & K_{P_{rz}} \\ 0 & -K_{S_{rz}} & K_{S_{ry}} \\ K_{S_{rz}} & 0 & -K_{S_{rx}} \\ -K_{S_{ry}} & K_{S_{rx}} & 0 \end{pmatrix} = i \begin{pmatrix} \mathbf{K}_{P_r} \cdot \\ \mathbf{K}_{S_r} \times \end{pmatrix} \tag{52}$$

$$(\Pi)_i^{-1} = \frac{-i}{\omega^2} \begin{pmatrix} \alpha_0^2 K_{P_{ix}} & 0 & \beta_0^2 K_{S_{iz}} & -\beta_0^2 K_{S_{iy}} \\ \alpha_0^2 K_{P_{iy}} & -\beta_0^2 K_{S_{iz}} & 0 & \beta_0^2 K_{S_{ix}} \\ \alpha_0^2 K_{P_{iz}} & \beta_0^2 K_{S_{iy}} & -\beta_0^2 K_{S_{ix}} & 0 \end{pmatrix} =$$

$$\frac{-i}{\omega^2}\left(\begin{array}{cc}\alpha_0^2\mathbf{K}_{P_i}^T\cdot & \beta_0^2(\mathbf{K}_{S_i}\times)^T\end{array}\right) \tag{53}$$

where subscripts r and i, respectively, denote the outgoing and incoming waves. Since the S-wave has two degree of freedom in apace, explicitly we can apply a transformation to reduce the number of components of S-waves to two. This can be done equivalently by the following pseudo-rotation matrix

$$E_j = \begin{pmatrix} 1 & \mathbf{0}^T \\ 0 & \mathbf{e}_{SI_j}{}^T \\ 0 & -\mathbf{e}_{SII_j}{}^T \end{pmatrix}, \qquad j = r, i \tag{54}$$

It can be directly checked that the following combination of Π and E operators gives the scattering matrix in $P - SI - SII$ bases

$$\mathcal{V}_{ve} = E_r \Pi_r \mathcal{V}_{ve} \Pi_i^{-1} E_i^{-1} =$$

$$\begin{pmatrix} (\alpha_0^2/\omega^2)\,\mathbf{K}_{P_r}^T V_{ve}\mathbf{K}_{Pi} & (\beta_0/\omega)\,\mathbf{K}_{P_r}^T V_{ve}\mathbf{e}_{S_{IIi}} & (\beta_0/\omega)\,\mathbf{K}_{P_r}^T V_{ve}\mathbf{e}_{S_{Ii}} \\ (\alpha_0^2/\omega\beta_0)\,\mathbf{e}_{SII_r}^T V_{ve}\mathbf{K}_{Pi} & \mathbf{e}_{SII_r}^T V_{ve}\mathbf{e}_{S_{IIi}} & \mathbf{e}_{SII_r}^T V_{ve}\mathbf{e}_{S_{Ii}} \\ (\alpha_0^2/\omega\beta_0)\,\mathbf{e}_{SI_r}^T V_{ve}\mathbf{K}_{Pi} & \mathbf{e}_{SI_r}^T V_{ve}\mathbf{e}_{S_{IIi}} & \mathbf{e}_{SI_r}^T V_{ve}\mathbf{e}_{S_{Ii}} \end{pmatrix} \tag{55}$$

ELEMENTS OF THE P-SI-SII SCATTERING MATRIX

To calculate the scattering matrix, we use the vectors $\mathbf{R}$ and $\mathbf{I}$ to indicate the reflected and incident vectors, respectively, which can be $\mathbf{K}_P$, $\mathbf{K}_S$, $\mathbf{e}_{SI}$ or $\mathbf{e}_{SII}$. Since the differential operators are sandwiched between unperturbed green functions, we replace the left derivatives with i multiplied by the reflected wavenumber vector $\mathbf{K}_r$ and right derivative with i multiplied by the incident wavenumber vector $\mathbf{K}_i$. For example if $\mathbf{R} = \mathbf{e}_{SII}$, the left derivative is replaced by $\mathbf{K}_{S_r}$. In the table (1)and (2) all possible choices for R and I wave types are presented.

R	$\mathbf{K}_P$	$\mathbf{K}_S$	$\mathbf{e}_{SI}$	$\mathbf{e}_{SII}$
∇^L	$i\mathbf{K}_{P_r}$	$i\mathbf{K}_{S_r}$	$i\mathbf{K}_{S_r}$	$i\mathbf{K}_{S_r}$

Table 1. Lefthand side derivative replacement

I	$\mathbf{K}_P$	$\mathbf{K}_S$	$\mathbf{e}_{SI}$	$\mathbf{e}_{SII}$
∇^R	$i\mathbf{K}_{P_i}$	$i\mathbf{K}_{S_i}$	$i\mathbf{K}_{S_i}$	$i\mathbf{K}_{S_i}$

Table 2. Righthand side derivative replacement

Lets write the scattering matrix element in the frequency independent form

$${}^R_I\mathbb{A}^J_{ve} = C^R_I \frac{\mathbf{R}^T V^J_{ve}\mathbf{I}}{A_J\omega^2}, \qquad J = \rho, \alpha_H, \beta_H, Q_{HP}, Q_{HS}. \tag{56}$$

Here C^R_I's are the factors multiplied in $\mathbf{R}^T V^J_{ve}\mathbf{I}$ terms in scattering potential, for instance $C^P_P = \alpha_0^2/\omega^2$. For perturbation terms we have

$${}^R_I\mathbb{A}^\rho_{ve} = \rho_0(\mathcal{F}^R_I - {}^R_I\mathcal{G}_\alpha - {}^R_I\mathcal{G}_\beta) \tag{57}$$

$$ {}^R_I\mathbb{A}_{ve}^{Q_{HP}} = \frac{i}{2} Q^{-1}_{HP_0} \left\{ {}^R_I\mathbb{A}_{ve}^{\alpha_H} \right\} = -i\rho_0 Q^{-1}_{HP_0} \mathcal{G}_\alpha \tag{58} $$

$$ {}^R_I\mathbb{A}_{ve}^{Q_{HS}} = \frac{i}{2} Q^{-1}_{HS_0} \left\{ {}^R_I\mathbb{A}_{ve}^{\beta_H} \right\} = -i\rho_0 Q^{-1}_{HS_0} \mathcal{G}_\beta \tag{59} $$

where we defined

$$ \mathcal{F}_I^R = C_I^R \mathbf{R}\cdot\mathbf{I} \tag{60} $$

$$ {}^R_I\mathcal{G}_\alpha = \frac{\alpha^2_{H_0}}{\omega^2} C_I^R (\mathbf{R}\cdot\mathbf{K}_r)(\mathbf{I}\cdot\mathbf{K}_i) \tag{61} $$

$$ {}^R_I\mathcal{G}_\beta = \frac{\beta^2_{H_0}}{\omega^2} C_I^R \left\{ (\mathbf{R}\cdot\mathbf{I})(\mathbf{K}_r\cdot\mathbf{K}_i) - 2(\mathbf{R}\cdot\mathbf{K}_r)(\mathbf{I}\cdot\mathbf{K}_i) + (\mathbf{I}\cdot\mathbf{K}_r)(\mathbf{R}\cdot\mathbf{K}_i) \right\} \tag{62} $$

So to determine each components of the scattering potential we need to calculate $\mathcal{F}_I^R$, ${}^R_I\mathcal{G}_\alpha$ and ${}^R_I\mathcal{G}_\beta$.

The P-P scattering potential

This element is the first element of the scattering matrix regarding to the case that a P-wave scattered to a P-wave. In this case incident and reflected waves are P-waves, $\mathbf{R} = \mathbf{K}_{Pr}$ and $\mathbf{I} = \mathbf{K}_{Pi}$, therefore we have

$$ \mathcal{F}_P^P = \frac{\alpha^2_{H_0}}{\omega^2} (1 + iQ^{-1}_{HP_0}) \mathbf{K}_{Pr}\cdot\mathbf{K}_{Pi} \tag{63} $$

$$ {}^P_P\mathcal{G}_\alpha = \frac{\alpha^4_{H_0}}{\omega^4} (1 + iQ^{-1}_{HP_0}) (\mathbf{K}_{Pr}\cdot\mathbf{K}_{Pr})(\mathbf{K}_{Pi}\cdot\mathbf{K}_{Pi}) \tag{64} $$

$$ {}^P_P\mathcal{G}_\beta = 2\frac{\beta^2_{H_0}\alpha^2_{H_0}}{\omega^4} (1 + iQ^{-1}_{HP_0}) (\mathbf{K}_{Pr}\times\mathbf{K}_{Pi})\cdot(\mathbf{K}_{Pr}\times\mathbf{K}_{Pi}) \tag{65} $$

The dot and cross products of wavenumber vectors can be expressed in terms of opening angle between the incident propagation vector $\mathbf{P}_{P_i}$ and reflected propagation vector $\mathbf{P}_{P_r}$ which is shown by σ_{PP} and the angles between the propagation and attenuation vectors. Using the identities in appendix B we arrive at

$$ \mathcal{F}_P^P = -\cos\sigma_{pp}(1 - iQ^{-1}_{HP_0}) + \frac{i}{2} Q^{-1}_{HP_0} \sin\sigma_{pp} \left(\tan\delta_{P_r} + \tan\delta_{P_i}\right) \tag{66} $$

$$ {}^P_P\mathcal{G}_\alpha = -(1 - iQ^{-1}_{HP_0}) \tag{67} $$

$$ {}^P_P\mathcal{G}_\beta = \left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2 \left\{ 2\sin^2\sigma_{pp}(1 - iQ^{-1}_{HP_0}) + iQ^{-1}_{HP_0} \sin 2\sigma_{pp} \left(\tan\delta_{P_r} - \tan\delta_{P_i}\right) \right\} \tag{68} $$

Here, δ_{P_r} and δ_{P_i} are the angle between the attenuation and propagation vectors for reflected and incident waves. Finally for PP element we have

$$ {}^P_P\mathbb{V}_{ve} = {}^P_P\mathbb{V}_e + iQ^{-1}_{HP_0}\rho_0 \left\{ \mathbb{F}^\rho A_\rho - 2A_{\alpha_H} + A_{Q_{HP}} + \mathbb{F}^{\beta_H} A_{\beta_H} \right\} - iQ^{-1}_{HS_0}\rho_0 \left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2 \sin^2\sigma A_{Q_{HS}} \tag{69} $$

where we define the following functions

$$\mathbb{F}^{\rho} = 1+\frac{1}{2}\sin\sigma_{pp}(\tan\delta_{P_r}+\tan\delta_{P_i})-\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\left\{2\sin^2\sigma_{pp}+\sin 2\sigma_{pp}(\tan\delta_{P_r}-\tan\delta_{P_i})\right\} \tag{70}$$

$$\mathbb{F}^{\beta_H} = \left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\left\{2\sin^2\sigma_{pp}+\sin 2\sigma_{pp}(\tan\delta_{P_r}-\tan\delta_{P_i})\right\} \tag{71}$$

and ${}^P_P\mathbb{V}_e$ is the elastic scattering matrix given by

$${}^P_P\mathbb{V}_e = -\rho_0\left\{1+\cos\sigma-2\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\sin^2\sigma\right\}A_\rho+2\rho_0 A_{\alpha_H}+2\rho_0\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\sin^2\sigma A_{\beta_H} \tag{72}$$

$P-SI$ scattering potential

This off-diagonal term represents the scattering of P-wave to a SI-wave. In this case incident wave number is given by $\mathbf{K}_{Pi}$ and reflected wavenumber by $\mathbf{K}_{Sr}$, $\mathbf{R}=\mathbf{e}_{SIr}$ and $\mathbf{I}=\mathbf{K}_{P_i}$, we have

$$\mathcal{F}^P_{SI} = \frac{\beta_{H_0}}{\omega}(1+\frac{i}{2}Q^{-1}_{HS_0})\mathbf{K}_{Pi}\cdot\mathbf{e}_{SI_r} \tag{73}$$

$${}^P_{SI}\mathcal{G}_\alpha = \frac{\alpha^2_{H_0}\beta_{H_0}}{\omega^3}(1+\frac{i}{2}Q^{-1}_{HS_0})(\mathbf{K}_{Pi}\cdot\mathbf{K}_{Pi})(\mathbf{e}_{SI_r}\cdot\mathbf{K}_{Si}) \tag{74}$$

$${}^P_{SI}\mathcal{G}_\beta = 2\frac{\beta^3_{H_0}}{\omega^3}(1+\frac{i}{2}Q^{-1}_{HS_0})(\mathbf{K}_{Pi}\cdot\mathbf{e}_{SI_r})(\mathbf{K}_{Pi}\cdot\mathbf{K}_{Si}) \tag{75}$$

Since $\mathbf{e}_{SI_r}$ vector is perpendicular to the wave number vector $\mathbf{K}_{S_r}$ so ${}^P_{SI}\mathcal{G}_\alpha=0$. As a result the terms of scattering element correspond to α_H and Q_{HP} are zero. Using the definition of $\mathbf{e}_{SI_r}$ we can write

$$\mathbf{K}_{P_i}\cdot\mathbf{e}_{SI_r} = \mathbf{K}_{P_i}\cdot(\frac{\mathbf{K}_{S_r}}{K_{S_r}}\times\mathbf{n}) = \frac{1}{K_{S_r}}(\mathbf{K}_{P_i}\times\mathbf{K}_{S_r})\cdot\mathbf{n} \tag{76}$$

since both $\mathbf{n}$ and $\mathbf{K}_{P_i}\times\mathbf{K}_{S_r}$ are in the $\mathbf{y}$-direction, we have

$$\mathbf{e}_{SI_r}\cdot\mathbf{K}_{P_i} = \frac{1}{K_{S_r}}|\mathbf{K}_{P_i}\times\mathbf{K}_{S_r}| = \frac{\beta_{H_0}}{\omega}|\mathbf{K}_{P_i}\times\mathbf{K}_{S_r}| \tag{77}$$

Inserting (77) in (73) and (75) and using the proper identities in appendix B, we get

$$\mathcal{F}^P_{SI} = \left(\frac{\beta_{H_0}}{2\alpha_{H_0}}\right)\times$$

$$\left\{2\sin\sigma_{ps}(1+iQ^{-1}_{HS_0})+iQ^{-1}_{HP_0}(\sin\sigma_{ps}-\cos\sigma_{ps}\tan\delta_{P_i})-iQ^{-1}_{HS_0}(\sin\sigma_{ps}-\cos\sigma_{ps}\tan\delta_{S_r})\right\} \tag{78}$$

$${}^P_P\mathcal{G}_\beta = \left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\sin 2\sigma_{ps}\left\{2+i(Q^{-1}_{HS_0}-Q^{-1}_{HP_0})\right\}$$

$$-i\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2 \sin^2\sigma_{ps}(Q^{-1}_{HP_0}\tan\delta_{P_i}+Q^{-1}_{HS_0}\tan\delta_{S_r})$$
$$+i\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2 Q^{-1}_{HP_0}(\sin\sigma_{ps}-\cos\sigma_{ps}\tan\delta_{P_i})$$
$$-i\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2 Q^{-1}_{HS_0}(\sin\sigma_{ps}-\cos\sigma_{ps}\tan\delta_{S_r})$$

$${}^{P}_{SI}\mathbb{V}_{ve}={}^{P}_{SI}\mathbb{V}_{e}+iQ^{-1}_{HP_0}\rho_0\left\{\mathbb{H}^{\rho}A_{\rho}+\mathbb{H}^{\beta}A_{\beta}\right\}+iQ^{-1}_{HS_0}\rho_0\left\{\mathbb{G}^{\rho}A_{\rho}+\mathbb{G}^{\beta}A_{\beta}-\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\sin 2\sigma A_{Q_{HS}}\right\} \tag{79}$$

where

$$\mathbb{G}^{\rho}=\left(\frac{\beta_{H_0}}{2\alpha_{H_0}}\right)(\sin\sigma_{ps}+\cos\sigma_{ps}\tan\delta_{S_r})+$$
$$\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\left(\sin 2\sigma_{ps}-\sin\sigma_{ps}+(\cos\sigma_{ps}-\sin^2\sigma_{ps})\tan\delta_{S_r}\right) \tag{80}$$
$$\mathbb{H}^{\rho}=\left(\frac{\beta_{H_0}}{2\alpha_{H_0}}\right)(\sin\sigma_{ps}-\cos\sigma_{ps}\tan\delta_{P_i})-$$
$$\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\left(\sin 2\sigma_{ps}-\sin\sigma_{ps}+(\cos\sigma_{ps}+\sin^2\sigma_{ps})\tan\delta_{P_i}\right) \tag{81}$$
$$\mathbb{G}^{\beta_H}=2\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\left(\sin 2\sigma_{ps}-\sin\sigma_{ps}+(\cos\sigma_{ps}-\sin^2\sigma_{ps})\tan\delta_{S_r}\right) \tag{82}$$
$$\mathbb{H}^{\beta_H}=2\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\left(\sin 2\sigma_{ps}-\sin\sigma_{ps}+(\cos\sigma_{ps}+\sin^2\sigma_{ps})\tan\delta_{P_i}\right) \tag{83}$$

and the elastic scattering matrix ${}^{P}_{SI}\mathbb{V}_{e}$ is given by

$${}^{P}_{SI}\mathbb{V}_{e}=\left\{\rho_0\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)\sin\sigma+\rho_0\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\sin 2\sigma\right\}A_{\rho}+\left\{2\rho_0\left(\frac{\beta_{H_0}}{\alpha_{H_0}}\right)^2\sin 2\sigma\right\}A_{\beta_H} \tag{84}$$

SI-TO-SI SCATTERING ELEMENT

The third diagonal element of the scattering matrix refers to scattering of SI-wave to SI-wave. In this case both incident and reflected waves are S-waves respectively represented by wave number vectors $\mathbf{K}_{S_i}$ and $\mathbf{K}_{S_r}$. For this element

$$\mathcal{F}^{SI}_{SI}=\mathbf{e}_{SI_r}\cdot\mathbf{e}_{SI_i} \tag{85}$$

$${}^{SI}_{SI}\mathcal{G}_{\beta}=\omega^{-2}\beta^2_{H_0}((\mathbf{e}_{SI_r}\cdot\mathbf{e}_{SI_i})(\mathbf{K}_{S_r}\cdot\mathbf{K}_{S_i})+(\mathbf{e}_{SI_i}\cdot\mathbf{K}_{S_r})(\mathbf{e}_{SI_r}\cdot\mathbf{K}_{S_i})) \tag{86}$$

First we note that

$$\mathbf{e}_{SI_r}\cdot\mathbf{e}_{SI_i}=(\hat{\mathbf{K}}_{S_r}\times\mathbf{n})\cdot(\hat{\mathbf{K}}_{S_i}\times\mathbf{n})=$$

$$(\hat{\mathbf{K}}_{S_r} \cdot \hat{\mathbf{K}}_{S_i})(\mathbf{n} \cdot \mathbf{n}) - (\hat{\mathbf{K}}_{S_r} \cdot \mathbf{n})(\hat{\mathbf{K}}_{S_i} \cdot \mathbf{n}) = \hat{\mathbf{K}}_{S_r} \cdot \hat{\mathbf{K}}_{S_i} \tag{87}$$

where we use the fact that $\mathbf{n}$ is perpendicular to wavenumber vectors. Also

$$\mathbf{e}_{SI_i} \cdot \mathbf{K}_{S_r} = (\hat{\mathbf{K}}_{S_i} \times \mathbf{n}) \cdot \mathbf{K}_{S_r} = (\mathbf{K}_{S_r} \times \hat{\mathbf{K}}_{S_i}) \cdot \mathbf{n} = \frac{1}{K_S}|\mathbf{K}_{S_r} \times \mathbf{K}_{S_i}| \tag{88}$$

$$\mathbf{e}_{SI_r} \cdot \mathbf{K}_{S_i} = (\hat{\mathbf{K}}_{S_r} \times \mathbf{n}) \cdot \mathbf{K}_{S_i} = (\mathbf{K}_{S_i} \times \hat{\mathbf{K}}_{S_r}) \cdot \mathbf{n} = -\frac{1}{K_S}|\mathbf{K}_{S_r} \times \mathbf{K}_{S_i}|. \tag{89}$$

Inserting (87)-(89) in (85) and (86) leads

$$\mathcal{F}_{SI}^{SI} = K_S^{-2}\mathbf{K}_{S_r} \cdot \mathbf{K}_{S_i} \tag{90}$$

$${}^{SI}_{SI}\mathcal{G}_\beta = \frac{\beta_{H_0}^2}{\omega^2}(1 + iQ_{HS_0}^{-1})(|\mathbf{K}_{S_r} \cdot \mathbf{K}_{S_i}|^2 - |\mathbf{K}_{S_r} \times \mathbf{K}_{S_i}|^2) \tag{91}$$

finally using the dot and cross products of wavenumber vectors we arrive at

$$\mathcal{F}_{SI}^{SI} = -\cos\sigma + \frac{i}{2}Q_{HS_0}^{-1}\sin\sigma\,(\tan\delta_{S_r} + \tan\delta_{S_i}) \tag{92}$$

$${}^{SI}_{SI}\mathcal{G}_\beta = \cos 2\sigma_{ss} - iQ_{HS_0}^{-1}\sin 2\sigma_{ss}\tan\delta_{S_r} \tag{93}$$

Now the related scattering element is

$${}^{SI}_{SI}\mathbb{V}_{ve} = {}^{SI}_{SI}\mathbb{V}_e + iQ_{HS_0}^{-1}\rho_0\left\{\mathbb{N}A_\rho + (\sin 2\sigma_{ss}\tan\delta_{S_r})A_\beta - \cos 2\sigma A_{Q_{HS}}\right\} \tag{94}$$

where

$$\mathbb{N} = \frac{1}{2}\sin\sigma\,(\tan\delta_{S_r} + \tan\delta_{S_i}) + \sin 2\sigma_{ss}\tan\delta_{S_r} \tag{95}$$

and the elastic scattering matrix is

$${}^{SI}_{SI}\mathbb{V}_e = -\left\{\rho_0(\cos\sigma + \cos 2\sigma_{ss})\right\}A_\rho - \left\{2\rho_0\cos 2\sigma_{ss}\right\}A_{\beta_H} \tag{96}$$

Scattering of SII-waves

The particle motion for SII-waves is a linear motion perpendicular to the plane that propagation and attenuation vectors constitute. It can be shown that SII-waves scattered only to SII-waves. Also P and SI waves do not converted to SII-waves. It means that scattering matrix elements correspond to $SII - P$, $P - SII$, $SII - SI$ and $SI - SII$ are zero. To analyse the scattering of SII-to-SII we obtain the related scattering matrix element, which is the second diagonal element of the scattering matrix

$${}^{SII}_{SII}\mathbb{V}_{ve} = {}^{SII}_{SII}\mathbb{V}_e + iQ_{HS_0}^{-1}\rho_0\left\{\mathbb{M}(A_\rho + 2A_\beta) + \cos\sigma A_{Q_{HS}}\right\} \tag{97}$$

where we defined

$$\mathbb{M} = -\cos\sigma - \frac{1}{2}\sin\sigma\,(\tan\delta_{S_r} + \tan\delta_{S_i}) \tag{98}$$

and the elastic scattering matrix is

$${}^{SII}_{SII}\mathbb{V}_e = \left\{\rho_0(1 + \cos\sigma)\right\}A_\rho + \left\{2\rho_0\cos\sigma\right\}A_{\beta_H} \tag{99}$$

VISCOELASTIC SENSITIVITIES

Gradient-based full-waveform inversion, is a method to estimate the subsurface parameters by iteratively minimizing the misfit function of the difference between recorded seismic data and modeled seismic data. Using the scattering potential obtained for viscoelastic medium we can compute the sensitivity of each scattered wave field to the perturbations. For example to calculate the sensitivity of PP element to Q_S we proceed as follow. The dependency of field data to the perturbation parameters is given by Stolt and Weglein (2012)

$$D(\mathbf{r}_g, \mathbf{r}_s, \omega) \approx \int_V d\mathbf{r}' G_L(\mathbf{r}_g, \mathbf{r}', \omega) \left\{ \sum_k C_k^L(\mathbf{r}', \omega) A_k(\mathbf{r}') C_k^R(\mathbf{r}', \omega) \right\} G_R(\mathbf{r}', \mathbf{r}_s, \omega), \tag{100}$$

Here $\mathbf{r}$ is the reflection point, $\mathbf{r}_s$ is the source position and $\mathbf{r}_g$ receiver position. Now we replace the left and right green functions with the P green functions and consider only to the terms related to $A_{Q_{HS}}$

$$\delta \mathrm{PP}_{ve}^{Q_S}(\mathbf{r}_g, \mathbf{r}_s, \omega) = i Q_{HS_0}^{-1} \delta \mathrm{Q}_{\mathrm{HS}}(\mathbf{r}_g, \mathbf{r}_s, \omega), \tag{101}$$

where

$$\delta \mathrm{Q}_{\mathrm{HS}}(\mathbf{r}_g, \mathbf{r}_s, \omega) = -\rho_0 \left(\frac{\beta_{H_0}}{\alpha_{H_0}} \right)^2 \sin^2 \sigma \int_V d\mathbf{r}' G_L^P(\mathbf{r}_g, \mathbf{r}', \omega) \delta A_{Q_{HS}}(\mathbf{r}') G_R^P(\mathbf{r}', \mathbf{r}_s, \omega) \tag{102}$$

Since the sensitivity is based on the response of scattered field data on small change at the point in space we set

$$\delta A_{Q_{HS}}(\mathbf{r}') = \delta A_{Q_{HS}} \delta(\mathbf{r} - \mathbf{r}') \tag{103}$$

and arrive to

$$\frac{\partial \mathrm{PP}_{ve}^{Q_S}(\mathbf{r}_g, \mathbf{r}_s, \omega)}{\partial A_{Q_{HS}}(\mathbf{r})} = -i Q_{HS_0}^{-1} \rho_0 \left(\frac{\beta_{H_0}}{\alpha_{H_0}} \right)^2 \sin^2 \sigma G_L^P(\mathbf{r}_g, \mathbf{r}, \omega) G_R^P(\mathbf{r}, \mathbf{r}_s, \omega) \tag{104}$$

SUMMARY AND CONCLUSION

The real earth has both attenuation and anisotropy, to get closer to the more realistic model of the earth we investigate the effects of viscosity in the context of scattering potential. In the viscoelastic medium because of attenuation waves generally are inhomogeneous. There are three types of waves that propagate in a viscoelastic medium. P and SI waves with elliptical motion in the plane of propagation and attenuation constitute and SII with the linear polarization perpendicular to that plane. To make a scattering potential with the elements refer to the scattering of viscoelastic wave modes, we construct the framework to transform the scattering potential from cartesian coordinates to the complex orthogonal coordinates constructed by three unit vectors in directions of P- and S-waves displacement vectors. The first is a complex unit vector in the direction of the wave vector. The second one is a complex unit vector in the direction of the displacement vector for SI wave and the last one is a unit vector in the direction of the displacement vector for SII wave. We have been shown that P and SI waves can only scattered to P and SI waves not SII and the SII wave can only reflected to SII waves. These results are consistent with the scattering of a general P, SI and SII waves on a viscoelastic boundary (Borcherdt, 2009).

The viscoelastic scattering potential we obtained has two term, the first is the elastic scattering potential and the second is a perturbation terms including the functions of open angle between incident and reflected propagation vectors and angle between propagation and attenuation vectors for incident and reflected waves. So that when attenuation goes to zero the elastic scattering potential is obtained.

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APPENDIX A: NOTES ON BIVECTORS

A bivector generally is represented by

$$\mathbf{a} = \mathbf{a}_1 + i\mathbf{a}_2 \tag{105}$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are real vectors. In terms of components of these vectors, a bivector can be written as

$$\mathbf{a} = a_x\mathbf{x} + a_y\mathbf{y} + a_z\mathbf{z}, \tag{106}$$

where $a_x = a_{1x} + ia_{2x}$ and so on. Two bivectors are equal if their real and imaginary parts be equal. Two bivectors are parallel if one of them be a product of other by a real or imaginary scalar. If a bivector is parallel to a real vector it has a real direction. Sum, dot and cross products of two bivectotrs $\mathbf{a} = \mathbf{a}_1 + i\mathbf{a}_2$ and $\mathbf{b} = \mathbf{b}_1 + i\mathbf{b}_2$ are given by

$$\mathbf{a} + \mathbf{b} = (\mathbf{a}_1 + \mathbf{a}_2) + i(\mathbf{b}_1 + \mathbf{b}_2) \tag{107}$$

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{a}_1 \cdot \mathbf{b}_1 - \mathbf{a}_2 \cdot \mathbf{b}_2) + i(\mathbf{a}_1 \cdot \mathbf{b}_2 + \mathbf{a}_2 \cdot \mathbf{b}_1) \tag{108}$$

$$\mathbf{a} \times \mathbf{b} = (\mathbf{a}_1 \times \mathbf{b}_1 - \mathbf{a}_2 \times \mathbf{b}_2) + i(\mathbf{a}_1 \times \mathbf{b}_2 + \mathbf{a}_2 \times \mathbf{b}_1) \tag{109}$$

In the other hand dot and cross product of a bivector by it's conjugate gives the different results

$$(\mathbf{a}_1 + i\mathbf{a}_2) \cdot (\mathbf{a}_1 - i\mathbf{a}_2) = \mathbf{a}_1 \cdot \mathbf{a}_1 + \mathbf{a}_2 \cdot \mathbf{a}_2 \tag{110}$$

$$(\mathbf{a}_1 + i\mathbf{a}_2) \times (\mathbf{a}_1 - i\mathbf{a}_2) = 2i\mathbf{a}_2 \times \mathbf{a}_1 \tag{111}$$

If we multiply a bivector by a phase factor $e^{i\varphi}$

$$\mathbf{a}' = \mathbf{a}_1' + i\mathbf{a}_2' = e^{i\varphi}(\mathbf{a}_1 + i\mathbf{a}_2) \tag{112}$$

it can be seen that the following expressions remain unchanged under this transformation

$$\mathbf{a}_1' \cdot \mathbf{a}_1' + \mathbf{a}_2' \cdot \mathbf{a}_2' = \mathbf{a}_1 \cdot \mathbf{a}_1 + \mathbf{a}_2 \cdot \mathbf{a}_2 \tag{113}$$

$$\mathbf{a}_2' \times \mathbf{a}_1' = \mathbf{a}_2 \times \mathbf{a}_1 \tag{114}$$

Following (112) we reach to

$$\mathbf{a}_1' = \mathbf{a}_1 \cos\varphi - \mathbf{a}_2 \sin\varphi \tag{115}$$

$$\mathbf{a}_2' = \mathbf{a}_2 \cos\varphi + \mathbf{a}_1 \sin\varphi \tag{116}$$

Assume that $\mathbf{a}_1$ and $\mathbf{a}_2$ are the pair of conjugate semi-diameters of a ellipse that named as the directional ellipse. Then $\mathbf{a}_1'$ and $\mathbf{a}_2'$ are also the pair of conjugate semi-diameters of that ellipse which is rotated from $\mathbf{a}_2$ to $\mathbf{a}_1$. As a result effect of multiplication of bivector by a phase factor appears as a rotation of ellipse to the angle of the phase factor (fig.2).

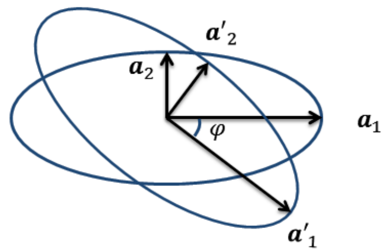

FIG. 2. Diagram illustrating the multiplication of bivector vector by a phase factor $e^{i\varphi}$.

If the bivector has a real direction, the ellipse degenerate to a straight line in the direction of bivector. As aresult any bivector can be written as a multiplication of a bivector and a cyclic factor as follows

$$\mathbf{c} + i\mathbf{d} = e^{i\varphi}(\mathbf{a} + i\mathbf{b}), \quad \mathbf{a}\cdot\mathbf{b} = 0 \tag{117}$$

product of $\mathbf{c} + i\mathbf{d}$ by itself gives

$$\mathbf{c}\cdot\mathbf{c} - \mathbf{d}\cdot\mathbf{d} + 2i\mathbf{c}\cdot\mathbf{d} = e^{2i\varphi}(\mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}) \tag{118}$$

which results

$$\tan 2\varphi = \frac{2\mathbf{c}\cdot\mathbf{d}}{\mathbf{c}\cdot\mathbf{c} - \mathbf{d}\cdot\mathbf{d}} \tag{119}$$

Assume that two bivectors

$$\mathbf{A} = e^{i\varphi_a}(\mathbf{a}_1 + i\mathbf{a}_2), \quad \mathbf{a}_1\cdot\mathbf{a}_2 = 0 \tag{120}$$

$$\mathbf{B} = e^{i\varphi_b}(\mathbf{b}_1 + i\mathbf{b}_2), \quad \mathbf{b}_1\cdot\mathbf{b}_2 = 0 \tag{121}$$

are perpendicular. Lets the two bivectors be coplanar

$$\mathbf{b}_1 = (\mathbf{b}_1\cdot\mathbf{a}_1)\mathbf{a}_1 + (\mathbf{b}_1\cdot\mathbf{a}_2)\mathbf{a}_2 \tag{122}$$

$$\mathbf{b}_2 = (\mathbf{b}_2\cdot\mathbf{a}_1)\mathbf{a}_1 + (\mathbf{b}_2\cdot\mathbf{a}_2)\mathbf{a}_2 \tag{123}$$

Now orthogonality condition, $\mathbf{A}\cdot\mathbf{B} = 0$, gives

$$(\mathbf{b}_1\cdot\mathbf{a}_1)(\mathbf{a}_1\cdot\mathbf{a}_1) = (\mathbf{b}_2\cdot\mathbf{a}_2)(\mathbf{a}_2\cdot\mathbf{a}_2) \tag{124}$$

$$(\mathbf{b}_1\cdot\mathbf{a}_2)(\mathbf{a}_2\cdot\mathbf{a}_2) = -(\mathbf{b}_2\cdot\mathbf{a}_1)(\mathbf{a}_1\cdot\mathbf{a}_1) \tag{125}$$

therefore we have

$$\mathbf{b} = Ae^{i\varphi_b}\left\{(\mathbf{a}_2\cdot\mathbf{a}_2)\mathbf{a}_1 + i(\mathbf{a}_1\cdot\mathbf{a}_1)\mathbf{a}_2\right\} \tag{126}$$

where

$$A = (\mathbf{b}_1\cdot\mathbf{a}_1)(\mathbf{a}_2\cdot\mathbf{a}_2) - i(\mathbf{b}_1\cdot\mathbf{a}_2)(\mathbf{a}_1\cdot\mathbf{a}_1) \tag{127}$$

So two perpendicular bivectors with the coincide plane have the similar elliptical plane and the major axes of the ellipses are perpendicular. If both vectors have real direction the problem reduces to the perpendicularity of the two directions.

APPENDIX B: LOW-LOSS VISCOELASTIC MEDIA

The phase speed for a homogeneous P-wave and S-wave are given by

$$\alpha_H = \frac{\omega}{|\mathbf{P}_P|} = 2\omega\left(\Re K_P^2\left[1+\sqrt{1+\Im K_P^2(\Re K_P^2)^{-1}}\right]\right)^{-1/2} \tag{128}$$

$$\beta_H = \frac{\omega}{|\mathbf{P}_S|} = 2\omega\left(\Re K_S^2\left[1+\sqrt{1+\Im K_S^2(\Re K_S^2)^{-1}}\right]\right)^{-1/2} \tag{129}$$

using the following relations

$$Q_{HP}^{-1} = -\frac{\Im k_P^2}{\Re k_P^2} = \frac{K_I + \frac{4}{3}M_I}{K_R + \frac{4}{3}M_R} \tag{130}$$

$$Q_{HS}^{-1} = -\frac{\Im k_S^2}{\Re k_S^2} = \frac{M_I}{M_R} \tag{131}$$

reduces to

$$\alpha_H = \frac{\omega}{|\mathbf{P}_P|} = \sqrt{\frac{K_R + \frac{4}{3}M_R}{\rho}\frac{2(1+Q_{HP}^{-2})}{1+\sqrt{1+Q_{HP}^{-2}}}} \tag{132}$$

$$\beta_H = \frac{\omega}{|\mathbf{P}_S|} = \sqrt{\frac{M_R}{\rho}\frac{2(1+Q_{HS}^{-2})}{1+\sqrt{1+Q_{HS}^{-2}}}} \tag{133}$$

for elastic media $Q_{HP}^{-1} = Q_{HS}^{-1} = 0$

$$\alpha_H = \alpha_e = \sqrt{\frac{K_R + \frac{4}{3}M_R}{\rho}} \tag{134}$$

$$\beta_H = \beta_e = \sqrt{\frac{M_R}{\rho}} \tag{135}$$

For P-wave and S-waves we have

$$K_P^2 = \frac{\rho\omega^2}{K + \frac{4}{3}M} = \frac{\rho\omega^2}{K_R + \frac{4}{3}M_R + i(K_I + \frac{4}{3}M_I)} \tag{136}$$

$$K_S^2 = \frac{\rho\omega^2}{M} = \frac{\rho\omega^2}{M_R + iM_I} \tag{137}$$

Using (130) and (131) we arrive at

$$K_P^2 = \frac{2\omega^2}{\alpha_H^2}\frac{1 - iQ_{HP}^{-1}}{1+\sqrt{1+Q_{HP}^{-2}}} \tag{138}$$

$$K_S^2 = \frac{2\omega^2}{\beta_H^2}\frac{1 - iQ_{HS}^{-1}}{1+\sqrt{1+Q_{HS}^{-2}}} \tag{139}$$

using the following identity

$$\sqrt{z} = \sqrt{\frac{|z| + z_R}{2}} + i\text{sing}[z_I]\sqrt{\frac{|z| - z_R}{2}} \tag{140}$$

we find that

$$K_P = \frac{\omega}{\alpha_H}\left(1 - \frac{iQ_{HP}^{-1}}{1 + \sqrt{1 + Q_{HP}^{-2}}}\right) \tag{141}$$

$$K_S = \frac{\omega}{\beta_H}\left(1 - \frac{iQ_{HS}^{-1}}{1 + \sqrt{1 + Q_{HS}^{-2}}}\right) \tag{142}$$

Materials with small amounts of absorbtion are low-loss viscoelastic media. In this materials $Q_{HP}^{-1}, Q_{HS}^{-1} \ll 1$. In this case

$$K_P = \frac{\omega}{\alpha_H}\left(1 - i\frac{Q_{HP}^{-1}}{2}\right) \tag{143}$$

$$K_S = \frac{\omega}{\beta_H}\left(1 - i\frac{Q_{HS}^{-1}}{2}\right) \tag{144}$$

For low-loss viscoelastic media we define the attenuation and propagation vectors for incident and reflected P-waves

$$\mathbf{P}_{P_r} = \frac{\omega}{\alpha_H}(\mathbf{x}\sin\theta_{P_r} + \mathbf{z}\cos\theta_{P_r}) \tag{145}$$

$$\mathbf{P}_{P_i} = \frac{\omega}{\alpha_H}(\mathbf{x}\sin\theta_{P_i} - \mathbf{z}\cos\theta_{P_i}) \tag{146}$$

$$\mathbf{P}_{S_r} = \frac{\omega}{\beta_H}(\mathbf{x}\sin\theta_{S_r} + \mathbf{z}\cos\theta_{S_r}) \tag{147}$$

$$\mathbf{P}_{S_i} = \frac{\omega}{\beta_H}(\mathbf{x}\sin\theta_{S_i} - \mathbf{z}\cos\theta_{S_i}) \tag{148}$$

$$\mathbf{A}_{P_r} = \frac{\omega Q_{HP}^{-1}\sec(\delta_{P_r})}{2\alpha_H}(\mathbf{x}\sin(\theta_{P_r} - \delta_{P_r}) + \mathbf{z}\cos(\theta_{P_r} - \delta_{P_r})) \tag{149}$$

$$\mathbf{A}_{P_i} = \frac{\omega Q_{HP}^{-1}\sec(\delta_{P_i})}{2\alpha_H}(\mathbf{x}\sin(\theta_{P_i} - \delta_{P_i}) - \mathbf{z}\cos(\theta_{P_i} - \delta_{P_i})) \tag{150}$$

$$\mathbf{A}_{S_r} = \frac{\omega Q_{HS}^{-1}\sec(\delta_{S_r})}{2\beta_H}(\mathbf{x}\sin(\theta_{S_r} - \delta_{S_r}) + \mathbf{z}\cos(\theta_{S_r} - \delta_{S_r})) \tag{151}$$

$$\mathbf{A}_{S_i} = \frac{\omega Q_{HS}^{-1}\sec(\delta_{S_i})}{2\beta_H}(\mathbf{x}\sin(\theta_{S_i} - \delta_{S_i}) - \mathbf{z}\cos(\theta_{S_i} - \delta_{S_i})) \tag{152}$$

The dot products of various types of propagation and attenuation vectors are

$$\mathbf{P}_{P_r} \cdot \mathbf{P}_{P_i} = -\frac{\omega^2}{\alpha_H^2}\cos\sigma_{pp} \tag{153}$$

$$\mathbf{P}_{P_r} \cdot \mathbf{P}_{S_i} = -\frac{\omega^2}{\alpha_H \beta_H} \cos\sigma_{ps} \tag{154}$$

$$\mathbf{P}_{S_r} \cdot \mathbf{P}_{S_i} = -\frac{\omega^2}{\beta_H^2} \cos\sigma_{ss} \tag{155}$$

$$\mathbf{A}_{P_r} \cdot \mathbf{A}_{P_i} \propto Q_{HP}^{-2} \approx 0 \tag{156}$$

$$\mathbf{A}_{S_r} \cdot \mathbf{A}_{S_i} \propto Q_{HS}^{-2} \approx 0 \tag{157}$$

$$\mathbf{A}_{P_r} \cdot \mathbf{A}_{S_i} \propto Q_{HP}^{-1} Q_{HS}^{-1} \approx 0 \tag{158}$$

$$\mathbf{P}_{P_r} \cdot \mathbf{A}_{P_i} = -\frac{\omega^2 Q_{HP}^{-1}}{2\alpha_H^2} (\cos\sigma_{pp} + \sin\sigma_{pp} \tan\delta_{P_i}) \tag{159}$$

$$\mathbf{P}_{P_i} \cdot \mathbf{A}_{P_r} = -\frac{\omega^2 Q_{HP}^{-1}}{2\alpha_H^2} (\cos\sigma_{pp} + \sin\sigma_{pp} \tan\delta_{P_r}) \tag{160}$$

$$\mathbf{P}_{P_r} \cdot \mathbf{A}_{S_i} = -\frac{\omega^2 Q_{HS}^{-1}}{2\alpha_H \beta_H} (\cos\sigma_{ps} + \sin\sigma_{ps} \tan\delta_{S_i}) \tag{161}$$

$$\mathbf{P}_{S_i} \cdot \mathbf{A}_{S_r} = -\frac{\omega^2 Q_{HS}^{-1}}{2\beta_H^2} (\cos\sigma_{ss} + \sin\sigma_{ss} \tan\delta_{S_i}) \tag{162}$$

$$\mathbf{P}_{S_r} \cdot \mathbf{A}_{P_i} = -\frac{\omega^2 Q_{HP}^{-1}}{2\alpha_H \beta_H} (\cos\sigma_{sp} + \sin\sigma_{sp} \tan\delta_{P_i}) \tag{163}$$

The cross products of various types of propagation and attenuation vectors are

$$\mathbf{P}_{P_r} \times \mathbf{P}_{P_i} = \frac{\omega^2}{\alpha_H^2} \sin\sigma_{pp} \mathbf{y} \tag{164}$$

$$\mathbf{P}_{P_r} \times \mathbf{P}_{S_i} = \frac{\omega^2}{\alpha_H \beta_H} \sin\sigma_{ps} \mathbf{y} \tag{165}$$

$$\mathbf{P}_{S_r} \times \mathbf{P}_{S_i} = \frac{\omega^2}{\beta_H^2} \sin\sigma_{ss} \mathbf{y} \tag{166}$$

$$\mathbf{P}_{P_r} \times \mathbf{A}_{P_i} = \frac{\omega^2 Q_{HP}^{-1}}{2\alpha_H^2} (\sin\sigma_{pp} - \cos\sigma_{pp} \tan\delta_{P_i}) \mathbf{y} \tag{167}$$

$$\mathbf{P}_{P_i} \times \mathbf{A}_{P_r} = \frac{\omega^2 Q_{HP}^{-1}}{2\alpha_H^2} (\sin\sigma_{pp} - \cos\sigma_{pp} \tan\delta_{P_r}) \mathbf{y} \tag{168}$$

$$\mathbf{P}_{P_r} \times \mathbf{A}_{S_i} = \frac{\omega^2 Q_{HS}^{-1}}{2\alpha_H \beta_H} (\sin\sigma_{ps} - \cos\sigma_{ps} \tan\delta_{S_i}) \mathbf{y} \tag{169}$$

$$\mathbf{P}_{S_r} \times \mathbf{A}_{S_i} = \frac{\omega^2 Q_{HS}^{-1}}{2\beta_H^2} (\sin\sigma_{ss} - \cos\sigma_{ss} \tan\delta_{S_i}) \mathbf{y} \tag{170}$$

$$\mathbf{A}_{S_r} \times \mathbf{A}_{P_i} \propto Q_{HS}^{-1} Q_{HP}^{-1} \approx 0 \tag{171}$$

$$\mathbf{A}_{P_r} \times \mathbf{A}_{P_i} \propto Q_{HP}^{-2} \approx 0 \tag{172}$$

$$\mathbf{A}_{S_r} \times \mathbf{A}_{S_i} \propto Q_{HS}^{-2} \approx 0 \tag{173}$$

where $\sigma_{pp} = \theta_{P_r} - \theta_{P_i}$ and so on. Also we have

$$K_P^2 = \mathbf{K}_P \cdot \mathbf{K}_P \approx \frac{\omega^2}{\alpha_{H_0}^2}(1 - iQ_{HP_0}^{-1}) \tag{174}$$

$$K_S^2 = \mathbf{K}_S \cdot \mathbf{K}_S \approx \frac{\omega^2}{\beta_{H_0}^2}(1 - iQ_{HS_0}^{-1}) \tag{175}$$

$$K_P = \sqrt{\mathbf{K}_P \cdot \mathbf{K}_P} \approx \frac{\omega}{\alpha_{H_0}}\left(1 - i\frac{Q_{HP_0}^{-1}}{2}\right) \tag{176}$$

$$K_S = \sqrt{\mathbf{K}_S \cdot \mathbf{K}_S} \approx \frac{\omega}{\beta_{H_0}}\left(1 - i\frac{Q_{HS_0}^{-1}}{2}\right) \tag{177}$$

The dot-products of wavenumber vectors are

$$\mathbf{K}_{P_r} \cdot \mathbf{K}_{P_i} = -\frac{\omega^2}{2\alpha_{H_0}^2}\left\{2\cos\sigma_{pp}(1 - iQ_{HP_0}^{-1}) - iQ_{HP_0}^{-1}\sin\sigma_{pp}\left(\tan\delta_{P_r} + \tan\delta_{P_i}\right)\right\} \tag{178}$$

$$\mathbf{K}_{S_r} \cdot \mathbf{K}_{S_i} = -\frac{\omega^2}{2\beta_{H_0}^2}\left\{2\cos\sigma_{ss}(1 - iQ_{HS_0}^{-1}) - iQ_{HS_0}^{-1}\sin\sigma_{ss}\left(\tan\delta_{S_r} + \tan\delta_{S_i}\right)\right\} \tag{179}$$

$$\mathbf{K}_{P_r} \cdot \mathbf{K}_{S_i} = -\frac{\omega^2}{2\alpha_{H_0}\beta_{H_0}}\left\{\cos\sigma_{ps}\left[2 - i(Q_{HS_0}^{-1} + Q_{HP_0}^{-1})\right] - i\sin\sigma_{ps}(Q_{HP_0}^{-1}\tan\delta_{P_r} + Q_{HS_0}^{-1}\tan\delta_{S_i})\right\} \tag{180}$$

The cross-products of wavenumber vectors are

$$\mathbf{K}_{P_r} \times \mathbf{K}_{P_i} = \frac{\omega^2}{2\alpha_{H_0}^2}\left\{2\sin\sigma_{pp}(1 - iQ_{HP_0}^{-1}) + iQ_{HP_0}^{-1}\cos\sigma_{pp}\left(\tan\delta_{P_r} - \tan\delta_{P_i}\right)\right\}\mathbf{y} \tag{181}$$

$$\mathbf{K}_{S_r} \times \mathbf{K}_{S_i} = \frac{\omega^2}{2\beta_{H_0}^2}\left\{2\sin\sigma_{ss}(1 - iQ_{HS_0}^{-1}) + iQ_{HS_0}^{-1}\cos\sigma_{ss}\left(\tan\delta_{S_r} - \tan\delta_{S_i}\right)\right\}\mathbf{y} \tag{182}$$

$$\mathbf{K}_{P_r} \times \mathbf{K}_{S_i} = \left\{\sin\sigma_{ps}\left[2 - i(Q_{HS_0}^{-1} + Q_{HP_0}^{-1})\right] + i\cos\sigma_{ps}(Q_{HP_0}^{-1}\tan\delta_{P_r} - Q_{HS_0}^{-1}\tan\delta_{S_i})\right\}\mathbf{y} \tag{183}$$

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