AVO modelling of linearized Zoeppritz approximations

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ABSTRACT

The reflection coefficients are investigated for the various approximation on Zoeppritz equation and the results compared with the exact solution. There some deviations near critical angles that for larger layer contrasts and larger angle of incidence these deviation are significant. Also, the effect of γ parameter on forward modelling is investigated. When the contrasts between the layers of model is larger, the γ parameter has more influence on forward modelling.

INTRODUCTION

The Aki and Richards *(*Aki and Richards, 1980) investigated the structure and property of the Earth's interior by analyzing the propagation of elastic waves. From a given source, parts of the generated waves travel through the earth's interior and part of the wave travels under the surface. The body waves (P-waves and S-waves) (Stein and Wysession, 2002) are the elastic waves which propagate in the interior of the Earth and the surface waves travel close to the surface. Seismic exploration mainly focus on body waves which contain much information about deeper structure of the earth. The transformation of the seismic waves upon incident on two homogeneous isotropic elastic half spaces in welded contact at a plane interface is given by Zoeppritz equation (Aki and Richards, 1980). This equation is the base for AVO inversion as forward model.

ZOEPPRITZ EQUATION

Zoeppritz equation describe the amplitudes of the reflected and transmitted P-wave and S-waves which are incident at boundary between the two media. It is derived under the assumption that incident angle at the boundary is below a critical angle. The critical angle of incidence is determined by the velocity of the upper and lower layers.When we apply the elastic boundary conditions continuity of displacement through the interface, the the Zoeppritz equations are obtained (Innanen, 2015). We have

$$P\begin{bmatrix} R_{pp} \\ R_{ps} \\ T_{pp} \\ T_{ps} \end{bmatrix} = b_p \text{ and } S\begin{bmatrix} R_{SS} \\ R_{SP} \\ T_{SS} \\ T_{SP} \end{bmatrix} = b_S \tag{1}$$

where

$$P = \begin{bmatrix} -X & -\Lambda_B(X) & CX & \Lambda_D(X) \\ \Lambda_1(X) & -BX & \Lambda_C(X) & -DX \\ 2B^2X\Lambda_1(X) & B\Lambda^B(X) & 2AD^2XA_C(X) & AD\Lambda^D(X) \\ -\Lambda^B(X) & 2B^2X\Lambda_B(X) & AC\Lambda^D(X) & -2AD^2X\Lambda_D(X) \end{bmatrix} \tag{2}$$

and

$$S = \begin{bmatrix} -Y & -\Lambda_{B^{-1}}(Y) & FY & -\Lambda_E(Y) \\ -\Lambda_1(Y) & -B^{-1}Y & \Lambda_F(Y) & EY \\ -2Y\Lambda_1(Y) & B^{-1}\Lambda^1(Y) & 2AF^2YA_F(Y) & -AE\Lambda^F(Y) \\ \Lambda^1(Y) & 2Y\Lambda_{B^{-1}}(Y) & AF\Lambda^F(Y) & 2AF^2Y\Lambda_E(Y) \end{bmatrix} \quad (3)$$

The right-hand sides of Eq.1 are

$$b_P = \begin{bmatrix} X \\ \Lambda_1(X) \\ 2B^2X\Lambda_1(X) \\ \Lambda^B(X) \end{bmatrix}, \text{ and } b_S = \begin{bmatrix} Y \\ \Lambda_1(Y) \\ 2Y\Lambda_1(Y) \\ \Lambda_1(Y) \end{bmatrix} \quad (4)$$

We let $X = Y = sin\theta$, and

$$\Lambda_j(Z) \equiv \sqrt{1 - j^2Z^2}\,, \qquad \Lambda^j(Z) \equiv 1 - 2j^2Z^2 \quad (5)$$

By solving these equations for a down-going incident the reflected P-wave, reflected S-wave, transmitted P-wave and transmitted S-wave are obtained. For AVO analysis, the R_{PP} and R_{PS} are the most exploited reflection coefficients i.e in typical seismic experiment using compressional wave sources, P-waves, and receiving the P-wave component and/ or S-wave component by receivers.

APPROXIMATION OF THE ZOEPPRITZ EQUATIONS

Zoeppritz's equations completely determine amplitudes of reflected and transmitted plane waves for all incidence angles. In order to gain more insight into the factors that control amplitude changes with angle/offset, and simplify computations, linearized approximations to the Zoeppritz equations have been developed. There are several approximation that we want to consider them and compare their results.

Exact solution

The exact PP wave reflection coefficients (Aki and Richards, 1980) can be written as

$$R_{PP} = \left[\left(b\frac{cos\theta_i}{V_{P0}} - c\frac{cos\theta_t}{V_{P1}}\right)F - \left(a + d\frac{cos\theta_i}{V_{P0}}\frac{cos\theta_t}{V_{P1}}\right)HP^2\right] / D \quad (6)$$

Where

$$a = \rho_1(1 - 2V_{S1}P^2) - \rho_0\left((1 - 2V_{S0}P^2)\right) \quad (7)$$

$$b = \rho_1(1 - 2V_{S1}P^2) + 2\rho_0V_{S0}P^2$$

$$c = \rho_1(1 - 2V_{S0}P^2) + 2\rho_1V_{S1}P^2$$

$$d = 2(\rho_1V_{S1} - \rho_0V_{S0}^2)$$

$$F = b\frac{cos\varphi_r}{V_{S0}} + c\frac{cos\varphi_t}{V_{S1}}$$

$$H = a - d\frac{cos\theta_t}{V_{P1}}\frac{cos\varphi_r}{V_{S0}}$$

$$D = \frac{\det P}{V_{P0}V_{P1}V_{S0}V_{S1}}$$

This equation shows the exact transformation of the reflection coefficient is a non-linear. For this reason, linearized approximations to the Zoeppritz equations have been developed.

Bortfeld's approximation

The Bortfield's approximation was revisited by Aki and Richards (Aki and Richards, 1980). The Aki and Richards and Frasier approximation is appealing because it is written as three terms,the first involving P-wave velocity, the second involving density, and the third involving S-wave velocity. Their formula can be written:

$$R_{PP}(\theta_i) \approx \frac{1}{2}(1 + tan^2\theta)\frac{\Delta V_P}{V_P} - 4\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta\frac{\Delta V_S}{V_S} + \frac{1}{2}[1 - 4\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta]\frac{\Delta\rho}{\rho} \quad (8)$$

where

$$\Delta V_P = V_{P1} - V_{P0} \quad (9)$$

$$\Delta V_S = V_{S1} - V_{S0}$$

$$\Delta\rho = \rho_1 - \rho_0$$

$$\theta_t = sin^{-1}(\frac{V_{P1}}{V_{P0}}\sin(\theta_i))$$

$$\theta = \frac{1}{2}(\theta_i + \theta_t)$$

This approximation is linear in three model parameters, the P-wave reflectivity, S-wave reflectivity, and density reflectivity, and it is even the starting equation for other approximations.

Shuey's approximation

The Shuey (Shuey, 1985) introduce the Poisson's ratio as parameter instead of the S-wave velocity and recalculated the Aki and Richards's approximation. Whereas the approximation in equation (Fatti et al., 1994) involved V_P, V_S and ρ, Shuey is obtained the new approximation of the Zoeppritz equations which involved V_P, ρ and σ, or Poisson's ratio. This approximation is obtained as

$$R_{PP} = R_P + R_h sin^2\theta + \frac{1}{2}\frac{\Delta V_P}{V_P}(tan^2\theta - sin^2\theta) \qquad (10)$$

Where

$$R_P = \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right) \qquad (11)$$

$$R_h = \left(R_P A_0 + \frac{\Delta\sigma}{(1-\sigma)^2}\right)$$

$$A_0 = B - 2(1+B)\frac{1-2\sigma}{1-\sigma}$$

$$\sigma = \frac{\frac{1}{\gamma^2}-2}{2(\frac{1}{\gamma^2}-1)}\ , \qquad \gamma = \frac{V_{S0}}{V_{P0}}$$

$$B = \frac{2\frac{\Delta V_P}{V_P}}{R_P}$$

$$\Delta\sigma = \sigma_1 - \sigma_0$$

In this approximation the first term gives the amplitude at normal incidence, the second term characterizes $R_{PP}(\theta)$ at intermediate angles, and the third term describes the approach to the critical angle. They noticed that when the ratio of β/α is 0.5 and the terms below 30° are dropped, then the Shuey's approximation can be reduced to two terms, a normal incidence reflectivity term, and a far-offset reflectivity term. This approximation is very useful if the analysis is in terms of the Poisson's ratios. It is an interesting property of rocks which is directly related to their elasticity.

Fatti's approximation

In 1994, Fatti et al. (Fatti et al., 1994) improved the Geo-stack method by incorporating the density changes instead of using the empirical relationship between V_P and ρ. By expressing the Aki and Richards`s approximation in terms of P-wave impedance, S-wave impedance, and density reflectivity the Fatti approximation is obtained as

$$R_{PP}(\theta_i) = \frac{1}{2}(1+tan^2\theta)\frac{\Delta I_P}{I_P} - 4\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta\frac{\Delta I_S}{I_S} - \frac{1}{2}\left(tan^2\theta - 4\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta\right)\frac{\Delta\rho}{\rho} \qquad (12)$$

Where $\frac{\Delta I_P}{I_P}$ and $\frac{\Delta I_S}{I_S}$ are

$$\frac{\Delta I_P}{I_P} = \frac{V_{P1}\rho_1 - V_{P0}\rho_0}{V_{P1}\rho_1 + V_{P0}\rho_0} \qquad (13)$$

And

$$\frac{\Delta I_S}{I_S} = \frac{V_{S1}\rho_1 - V_{S0}\rho_0}{V_{S1}\rho_1 + V_{S0}\rho_0} \qquad (14)$$

For angle of incidence less than 35 degrees and α/β ratio between 1.5 and 2.0, the above equation simplifies to

$$R_{PP}(\theta_i) = \frac{1}{2}(1 + tan^2\theta)\frac{\Delta I_P}{I_P} - 4\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta\frac{\Delta I_S}{I_S} \qquad (15)$$

Verm and Hilterman's approximation

It is essentially the same as the Shuey's equation, with the simplification that the third term is ignored, since it is vanishingly small for small offsets (being the difference between the squared sine and squared tangent of small angles). Therefor Hilterman (Verm and Hilterman, 1995) simplified Shuey's equation even further by making the following assumptions: (a) Use only the first two terms of Shuey's equatin , and (b) Set $A_0 = -1$. Then, equation (Shuey, 1985) simplifies to:

$$R_{PP} = R_P + R_h sin^2\theta \qquad (16)$$

Where

$$R_h = \left(R_P A_0 + \frac{\Delta\sigma}{(1-\sigma)^2}\right) = (-R_P + \frac{\Delta\sigma}{(1-\sigma)^2}) \qquad (17)$$

Substitute Eq. 13 into Eq. 12. We have

$$R_{PP} = R_P + R_h sin^2\theta = Rp + \left(-R_P + \frac{\Delta\sigma}{(1-\sigma)^2}\right) sin^2\theta \qquad (18)$$

$$= Rp(1 - sin^2\theta) + \frac{\Delta\sigma}{(1-\sigma)^2} sin^2\theta$$

This equation also can be written as

$$Rpp = Rp + [\frac{9}{4\Delta\sigma} - R_P]sin^2\theta \qquad (19)$$

Where

$$G = \frac{9}{4}\Delta\sigma - R_P \qquad (20)$$

Therefore, the $R_{PP}(\theta)$ has the form

$$R_{PP} = R_P + G sin^2\theta \qquad (21)$$

Where R_P gives the intercept and G is the AVO gradient (slope) obtained by performing a linear regression analysis on the seismic amplitudes. This equation is linear if we plot R as a function of $sin2\theta$. We could then perform a linear regression analysis on the seismic amplitudes to come up with estimates of both intercept R_P, and gradient G.

Smith and Gidlow's approximation

The new approximation based on the Aki and Richards's equation was given by Smith and Gidlow (Smith and Gidlow, 2003). They used this approximation to perform a weighted stack on the corrected seismic gathers to produce information about the rock properties of reservoirs. Smith and Gidlow rearranged the Aki and Richards's approximation as

$$R_{PP}(\theta_i) = \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{\Delta\rho}{\rho}\right) - 2\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta\left(2\frac{\Delta V_S}{V_S} + \frac{\Delta\rho}{\rho}\right) + \frac{1}{2}tan^2\theta\frac{\Delta V_P}{V_P} \quad (22)$$

When eliminating density by applying Gardner's relation

$$\rho = aV_P^b = a(V_P)^{1/4} \quad (23)$$

And

$$\frac{\Delta\rho}{\rho} = b\frac{\Delta V_P}{V_P} = \frac{1}{4}\frac{\Delta V_P}{V_P} \quad (24)$$

Where a and b are empirical constants which are directly related to the type of rock.

Substituting Eq.20 into Eq.18 gives:

$$R_{PP}(\theta_i) = \frac{1}{2}\left(\frac{\Delta V_P}{V_P} + \frac{1}{4}\frac{\Delta V_P}{V_P}\right) - 2\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta\left(2\frac{\Delta V_S}{V_S} + \frac{1}{4}\frac{\Delta V_P}{V_P}\right) + \frac{1}{2}tan^2\theta\frac{\Delta V_P}{V_P} \quad (25)$$

This equation also can be written as

$$R_{PP}(\theta_i) = \frac{1}{2}(1 + a + tan^2\theta - 4a\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta)\frac{\Delta V_P}{V_P} - 4\left(\frac{V_{S0}}{V_{P0}}\right)^2 sin^2\theta\frac{\Delta V_S}{V_S} \quad (26)$$

Where a=0.25 is constant in Gardner`s equation. This simplification allowed them to obtain estimates of rock properties by using a weighted stacking method.

RESULTS AND DISCUSSIONS

The results of different approximation are compared to the exact equation. The geologic parameters are given in Table 1. The reflection coefficients are calculated and plotted against the angle of incidence up to the critical angle. The various approximations are compared to the exact Zoeppritz equation for small layer contrast (model 1 and 2) and large layer contrast (model 3 and 4). Figure 1 shows the comparison of the various approximation for small layer contrast (model 1). We can see that all the approximations are in a good agreement with the exact equation up to angle range from 50°. After this angle, there are some deviations near the critical angle particularly for Verm and Hilterman's equation. Our results show that the Shuy's, Aki and Richards's and Smith and Gidlow's approximations are in a goog agreement with the exact equation up to the critical angle. Figure 2 shows the comparison of the various approximation for small layer contrast (model 2). The results show that all the approximations are in a good agreement with the exact solution up to angle from 30° to 40°. In the large angles some deviations are seen near the critical angle. The Shuey's and Verm and Hilterman approximations have some deviations near critical angle, while the Fatti's approximation is in a good agreement with exact solution up to 50°. The other approximations (Aki and Richards's and Smith and Gidlow's) have same results with the exact equation up to the critical angle. Similarly, the reflection coefficients for large layer contrast are investigated. The results show that there are some significant deviation. Figure 3 the comparison of the various approximation for large layer contrast (model 3). Aki and Richards's and Smith and Gidlow's approximation are a good agreement with each other but show some deviations from the exact solution. The Shuey's approximation and the Fatti's approximation are in a good agreement up to only 25° to 35° respectively. Figure 4 shows comparison of the various approximations for model 4. The deviations become larger as compared to other models. We can see that the reflection coefficient for different model has different behaviour for each approximation. For large layer contrast (large angle of incidence) the deviation from exact is increased.

Table 1. The geologic parameters for four models.

Model	$V_{P0}(m/s)$	$V_{S0}(m/s)$	$\rho_0(kg/m^3)$	$V_{P1}(m/s)$	$V_{S1}(m/s)$	$\rho_1(kg/m^3)$
1	3000	1800	2200	3100	1900	2250
2	3000	1800	2200	3200	2000	2250
3	3000	1800	2200	4000	2500	2400
3	3000	1800	2200	4500	3000	2600

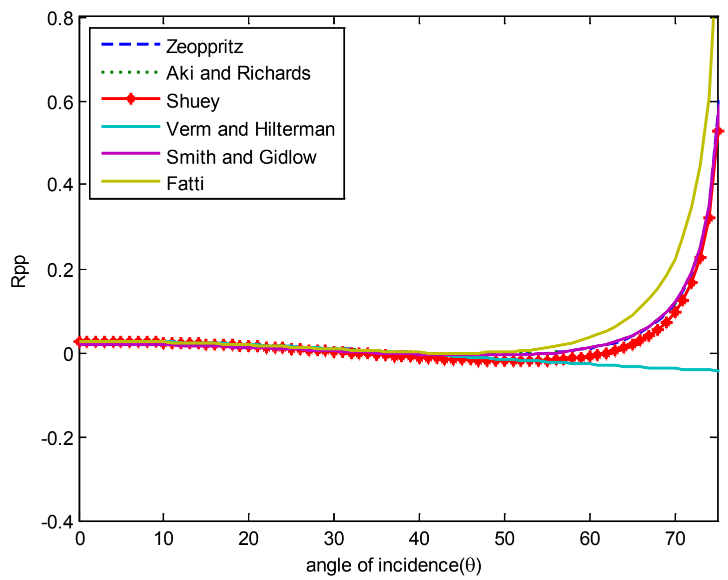

FIG.1. Illustrate the reflection coefficients (Rpp) for different approximations using model 1 in Table 1.

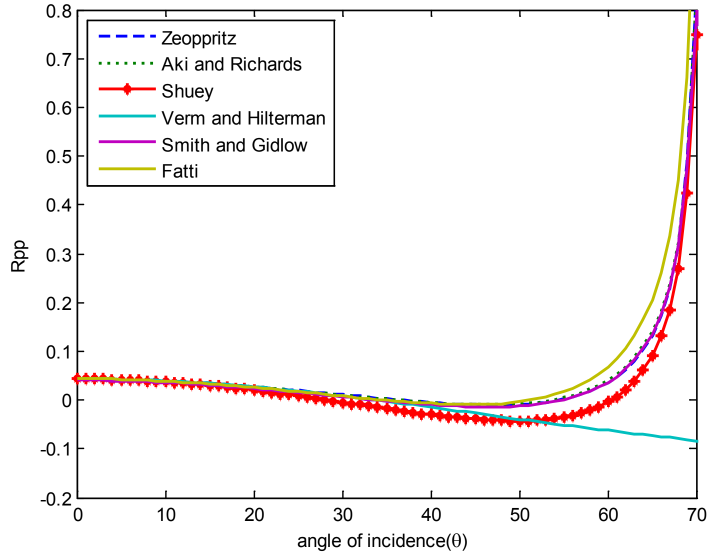

FIG. 2. Illustrate the reflection coefficients (Rpp) for different approximations using model 2 in Table 1.

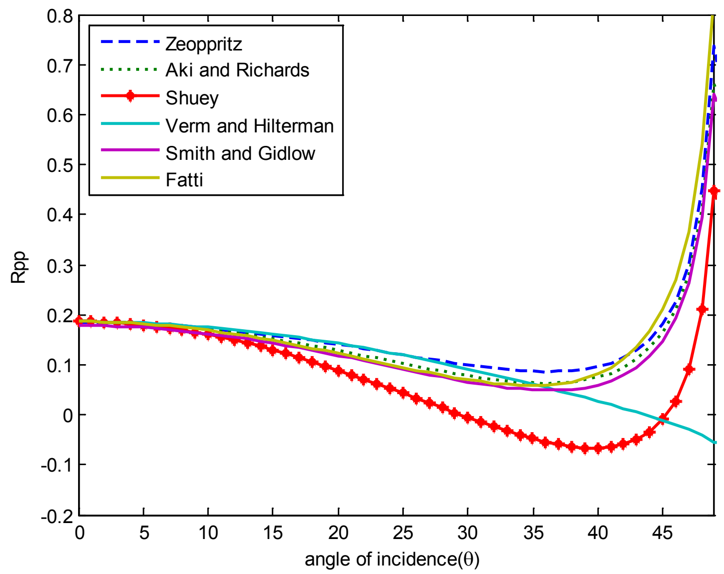

FIG. 3. Illustrate the reflection coefficients (Rpp) for different approximations using model 3 in Table 1.

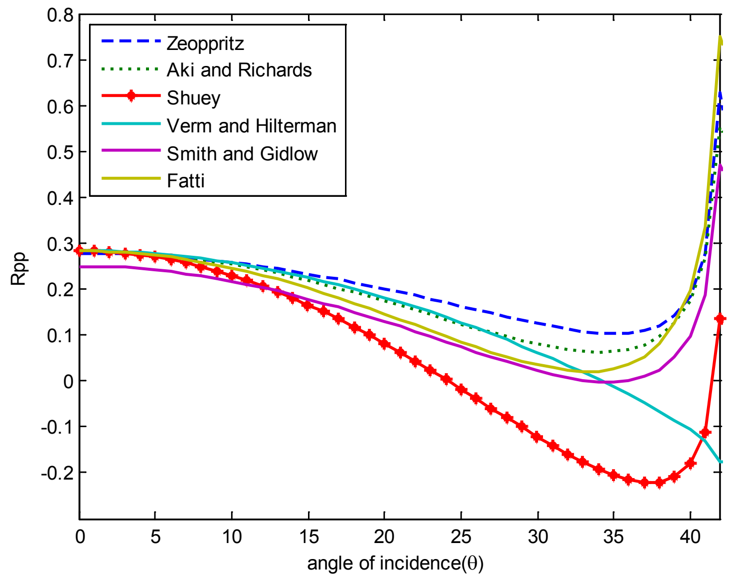

FIG. 4. Illustrate the reflection coefficients (Rpp) for different approximations using model 3 in Table 1.

The γ parameter (the S-wave velocity to P-wave velocity ratio) is a part of all approximation. So, we consider the effect of this parameter in forward modeling. To see this effect the reflection coefficient are calculated for different value of γ. Figure 5 show the reflection coefficient using Zoeppritz equation and Aki and Richards's approximation for actual value of γ and four different values of γ which are difference from the actual value (model 1). The results show that for the actual value of γ the Aki and Richards's approximation is a good agreement with the exact solution, while the wrong values of γ lead to small deviation from the exact solution. Similarly, the effect of γ on the forward modeling for other models (model 2, 3, and 4) are investigated. The results show that the deviations become larger as compared to model 1 (figures 6 7, and 8). We can see that when the contrast between the layer model parameter is increased, the parameter γ has more influence on forward modeling.

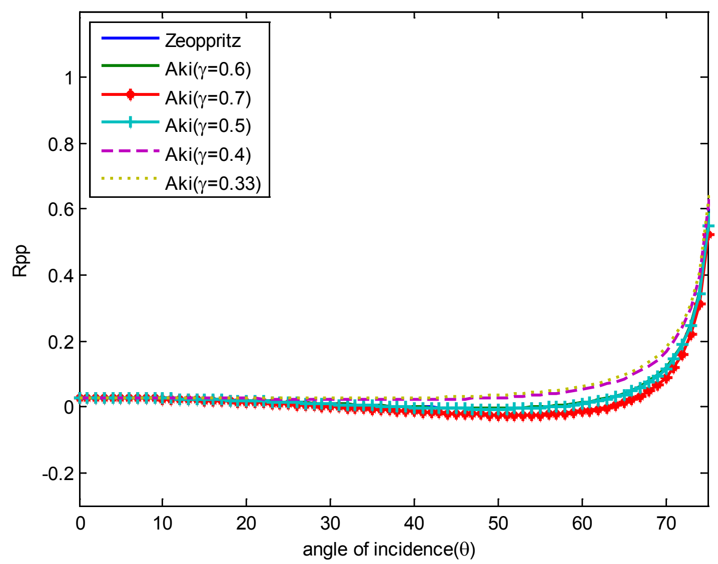

Fig. 5. Illustrate the effect of γ on forward model using model 1 for $\gamma = 0.6$, $\gamma = 033$, $\gamma = 0.4$, $\gamma = 0.5$, and $\gamma = 07$.

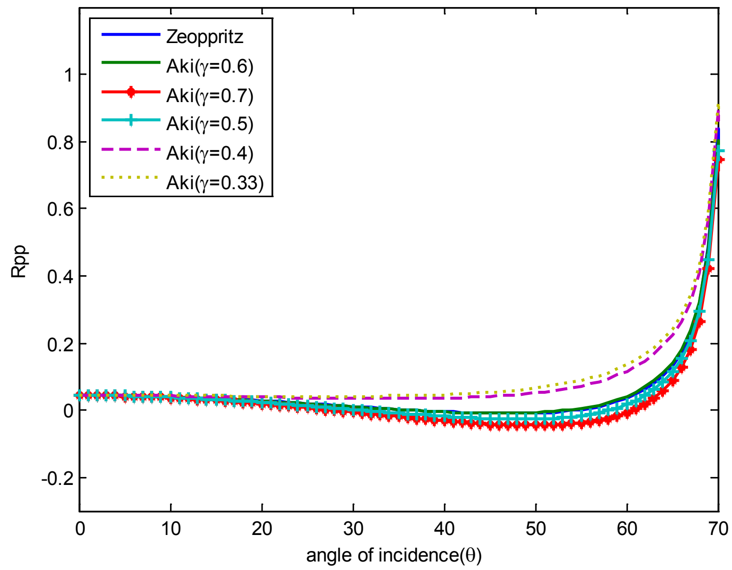

FIG. 6. The effect of γ on forward model using model 2 for $\gamma = 0.6$, $\gamma = 033, \gamma = 0.4, \gamma = 0.5$, and $\gamma = 07$.

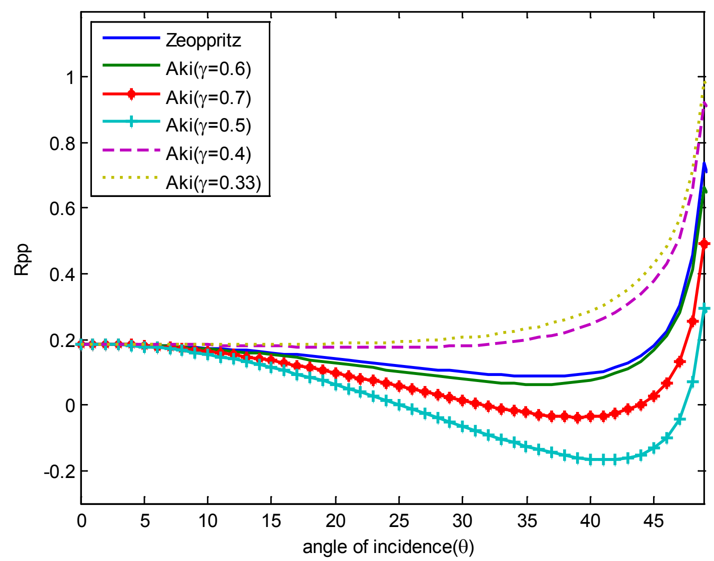

FIG. 7. Illustrate the effect of γ on forward model using model 3 for $\gamma = 0.6$, $\gamma = 033, \gamma = 0.4, \gamma = 0.5$, and $\gamma = 07$.

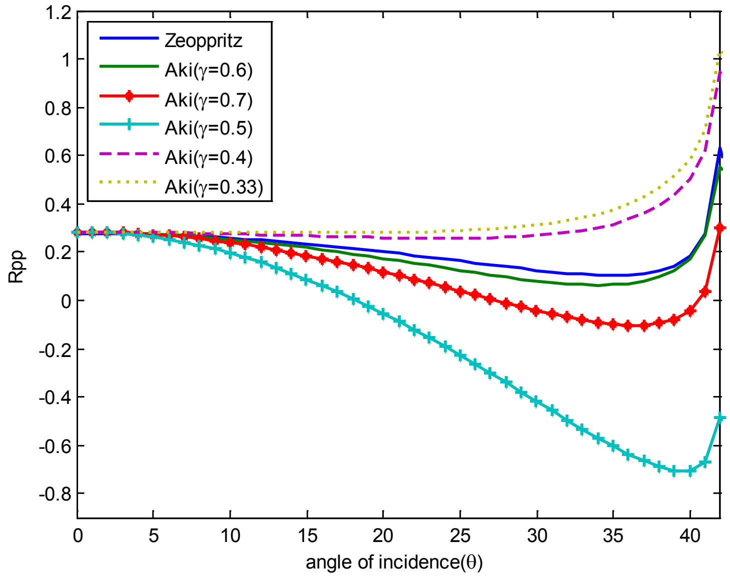

FIG. 8. Illustrate the effect of γ on forward model using model 4 for $\gamma = 0.6$, $\gamma = 033$, $\gamma = 0.4$, $\gamma = 0.5$, and $\gamma = 07$.

CONCLUSIONS

The exact solution of the Zoeppritz equations and the linear approximations are investigated. The reflection coefficient for the various approximations are calculated using four models which have small and larger layer contrasts. There are the larger the deviation is from the exact solution for the larger the angle of incidence (or the larger the layer contrasts). These deviation can be reasons to wrong results for inversion. In addition in this work the effect of γ on the forward modeling are investigated. When the contrast between the layer model parameter is increased, the parameter γ has more influence on forward modeling. Therefore, the act of actual value of velocities can reduce the effects of γ on forward modeling.

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