Absorption in FWI – some questions and answers

Kris Innanen

ABSTRACT

In this short note we cover two questions concerning the inclusion in seismic FWI of an attempt to solve simultaneously for elastic and anelastic geological properties, i.e., Q_P and Q_S as well as (say) V_P, V_S and ρ. The first question is whether or not to do so: anecdotally (and incorrectly), attenuation parameters have been suggested to be unimportant to FWI, since they dominate at high frequencies and FWI is primarily concerned with low. The second is really several questions: where in the mechanisms of FWI (e.g., within the machinery of the gradient and the various approximate Hessians), are the tasks we normally associate with Q contained? Is there a component of the Hessian operator tasked with Q compensation, for instance.

INTRODUCTION

In this note we formulate an an-acoustic FWI framework and use it to answer two questions about how such a procedure would work, and whether and when we should consider using it. The paper is meant to act as a companion to Innanen (2015), and has the same motivations and references as those in the introduction to that paper. Here, the question of whether or not Q is important to FWI is considered, which is also a key theme in the companion work.

The work is broken up into three parts. First, we frame FWI with an an-acoustic model, in which three parameters, P-wave velocity, quality factor Q, and dispersive reference frequency ω_r are treated as unknowns. Cases involving only attenuation, only dispersion, and a reduced two parameter (velocity and Q) problem, are also set up. The second component uses sensitivity forms to comment on the possibility that Q is unimportant to FWI provided FWI minds low frequencies. The third component uses the gradient and Hessian forms to comment on the mechanisms in a Gauss-Newton update by which something akin to Q compensation occurs.

FRAMING AN-ACOUSTIC FULL WAVEFORM INVERSION

We will define anacoustic wave propagation as that described by the equation

$$\mathcal{L}\, G(\mathbf{r}, \mathbf{r}_s) = \left[\nabla^2 + \omega^2 s(\mathbf{r})\right] G(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s), \tag{1}$$

where ω dependence of G is assumed, and the general model parameter s is

$$s(\mathbf{r}) = \frac{1}{c^2(\mathbf{r})} \left\{ 1 + \frac{1}{Q(\mathbf{r})} \left[i - \frac{2}{\pi} \log\left(\frac{\omega}{\omega_r(\mathbf{r})}\right)\right]\right\}, \tag{2}$$

that is, three basic anacoustic parameters c, Q and ω_r, corresponding to the parameters of a common nearly constant Q model (Aki and Richards, 2002). These parameters appear in the propagation constant in a curious hierarchy, which is more easily seen by writing s as

$$s(\mathbf{r}) = s_c(\mathbf{r}) + \gamma\, s_c(\mathbf{r}) s_q(\mathbf{r}) + \lambda\, s_c(\mathbf{r}) s_q(\mathbf{r}) s_\omega(\mathbf{r}), \tag{3}$$

where

$$s_c(\mathbf{r}) = \frac{1}{c^2(\mathbf{r})}, \quad s_q(\mathbf{r}) = \frac{1}{Q(\mathbf{r})}, \quad s_\omega(\mathbf{r}) = \log \omega_r(\mathbf{r}), \tag{4}$$

and

$$\gamma = i - \frac{2}{\pi} \log \omega, \quad \lambda = \frac{2}{\pi}. \tag{5}$$

The separability in particular of the reference frequency parameter from the Q parameter is far from certain at the outset, so we will also include a case where ω_r is a fixed quantity whose value is assumed or known a priori. Then, s breaks up as

$$s(\mathbf{r}) = s_c(\mathbf{r}) + \beta\, s_c(\mathbf{r}) s_q(\mathbf{r}), \tag{6}$$

where

$$\beta = i - \frac{2}{\pi} \log \left(\frac{\omega}{\omega_r}\right). \tag{7}$$

Finally, to support a later examination of attenuation vs. dispersion in the sensitivities, we introduce two convenient model parameters:

$$\begin{aligned} s_a(\mathbf{r}) &= \frac{1}{c^2(\mathbf{r})}\left[1 + \frac{i}{Q(\mathbf{r})}\right] = s_c(\mathbf{r}) + \mu s_c(\mathbf{r}) s_q(\mathbf{r}) \\ s_d(\mathbf{r}) &= \frac{1}{c^2(\mathbf{r})}\left[1 - \frac{2}{\pi Q(\mathbf{r})} \log\left(\frac{\omega}{\omega_r}\right)\right] = s_c(\mathbf{r}) + \nu s_c(\mathbf{r}) s_q(\mathbf{r}), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mu &= i \\ \nu &= -\frac{2}{\pi} \log\left(\frac{\omega}{\omega_r}\right). \end{aligned} \tag{9}$$

The first of these includes attenuation in the wave while neglecting dispersion, and the second includes dispersion while neglecting attenuation. Neither are causal; they are used as tools to examine the internal workings of FWI only, and are not recommended for use to form proper inverse procedures. Because we will use them later, let us also write down explicitly the equations and Green's functions we associate with these models:

$$\left[\nabla^2 + \omega^2 s_a(\mathbf{r})\right] G^a(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s), \tag{10}$$

and

$$\left[\nabla^2 + \omega^2 s_d(\mathbf{r})\right] G^d(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s), \tag{11}$$

respectively.

Anacoustic sensitivities

Multiparameter anacoustic sensitivities can be calculated based on any one of the four anacoustic scenarios described above, meaning equation (1) with s defined as in one of equations (3), (6), (10), or (11). As a note of caution, to keep the nomenclature under control we will re-use the same terms in each case below. When the results are used later the particular case being discussed will be explicitly clarified.

Case 1: Three parameter anacoustic media.

Beginning with the first of these four cases, we define background medium s_0:

$$s_0(\mathbf{r}) = s_{c_0}(\mathbf{r}) + \gamma\; s_{c_0}(\mathbf{r}) s_{q_0}(\mathbf{r}) + \lambda\; s_{c_0}(\mathbf{r}) s_{q_0}(\mathbf{r}) s_{\omega_0}(\mathbf{r}), \tag{12}$$

consider the wave G_0 to propagate within it, and then, one at a time, perturb the three model parameters to create slighty different waves. A small perturbation δs_c in the P-wave velocity c gives rise to the wave G_c satisfying

$$\mathcal{L}_0 G_c(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s) - \omega^2 \left[1 + s_{q_0}(\mathbf{r}) \left(\gamma + \lambda s_{\omega_0}(\mathbf{r})\right)\right] G_c(\mathbf{r}, \mathbf{r}_s) \delta s_c(\mathbf{r}); \tag{13}$$

a similar perturbation in Q gives rise to G_q satisfying

$$\mathcal{L}_0 G_q(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s) - \omega^2 \left[\left(\gamma + \lambda s_{\omega_0}(\mathbf{r})\right) s_{c_0}(\mathbf{r})\right] G_q(\mathbf{r}, \mathbf{r}_s) \delta s_q(\mathbf{r}); \tag{14}$$

and likewise a perturbation in the reference frequency gives rise to G_ω satisfying

$$\mathcal{L}_0 G_\omega(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s) - \omega^2 \left[\lambda s_{c_0}(\mathbf{r}) s_{q_0}(\mathbf{r})\right] G_\omega(\mathbf{r}, \mathbf{r}_s) \delta s_\omega(\mathbf{r}). \tag{15}$$

In all cases the operator $\mathcal{L}_0$ is the same one used in equation (1) but with s set to s_0. A linearized scattering equation can be derived for each of $\delta G_c = G_c - G_0$, $\delta G_q = G_q - G_0$ and $\delta G_\omega = G_\omega - G_0$. Using these, and by specifying each perturbation to be a delta function, the sensitivities for three parameters are found to be:

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_c(\mathbf{r})} = -\omega^2 \left[1 + s_{q_0}(\mathbf{r}) \left(\gamma + \lambda s_{\omega_0}(\mathbf{r})\right)\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s), \tag{16}$$

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_q(\mathbf{r})} = -\omega^2 \left[\left(\gamma + \lambda s_{\omega_0}(\mathbf{r})\right) s_{c_0}(\mathbf{r})\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s), \tag{17}$$

and

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_\omega(\mathbf{r})} = -\omega^2 \left[\lambda s_{c_0}(\mathbf{r}) s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s), \tag{18}$$

respectively.

Case 2: Two parameter anacoustic media.

Using equation (1) and the background medium definition

$$s_0(\mathbf{r}) = s_{c_0}(\mathbf{r}) + \beta\, s_{c_0}(\mathbf{r}) s_{q_0}(\mathbf{r}), \tag{19}$$

the two parameter sensitivities for c and Q are similarly found to be

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_c(\mathbf{r})} = -\omega^2 \left[1 + \beta s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s,), \tag{20}$$

and

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_q(\mathbf{r})} = -\omega^2 \left[\beta s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s), \tag{21}$$

respectively.

Case 3: Two parameter attenuation-only media.

The attenuation-only case has the same form as case 2, but with μ substituted for β. Thus, we have

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_c(\mathbf{r})} = -\omega^2 \left[1 + \mu s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s), \tag{22}$$

and

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_q(\mathbf{r})} = -\omega^2 \left[\mu s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s). \tag{23}$$

Case 4: Two parameter dispersion-only media.

Similarly, the dispersion-only case is the same as Cases 2 and 3, with ν used rather than β or μ:

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_c(\mathbf{r})} = -\omega^2 \left[1 + \nu s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s), \tag{24}$$

and

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_q(\mathbf{r})} = -\omega^2 \left[\nu s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s). \tag{25}$$

Two parameter an-acoustic gradients

Gradients of objective functions which are based on the sum of the squares of the residuals $\delta P(\mathbf{r}_g, \mathbf{r}_s)$, i.e., the difference between the observed and modelled data at a particular iteration, have a common form. If the sensitivities for parameter X are known, the gradient is simply

$$g_X(\mathbf{r}) = -\sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega \frac{\partial G(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_X(\mathbf{r})} \delta P^*(\mathbf{r}_g,\mathbf{r}_s), \tag{26}$$

where * indicates the complex conjugate. Thus, for the two parameter an-acoustic case (Case 2 above), the two gradients are

$$g_c(\mathbf{r}) = \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega\omega^2 \left[1 + \beta s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g,\mathbf{r}) G_0(\mathbf{r},\mathbf{r}_s) \delta P^*(\mathbf{r}_g,\mathbf{r}_s), \tag{27}$$

and

$$g_q(\mathbf{r}) = \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega\omega^2 \left[\beta s_{q_0}(\mathbf{r})\right] G_0(\mathbf{r}_g,\mathbf{r}) G_0(\mathbf{r},\mathbf{r}_s) \delta P^*(\mathbf{r}_g,\mathbf{r}_s). \tag{28}$$

From other multiparameter FWI analyses we expect that such gradients will be strongly affected by parameter cross-talk, however, and indeed on first analysis this seems likely. The residuals in both gradients are weighted to accentuate the influence of the two parameters, through the β terms, the squared frequency, and the Green's functions, however no accommodation is made for the fact that δP^* is co-determined by two parameters.

An-acoustic Gauss-Newton Hessian operators

The Hessian operator, as approximated in a Gauss-Newton approach, in the two parameter an-acoustic approximation will involve four elements and a 2×2 system. Defining for convenience

$$\mathcal{G}(\mathbf{r}_g,\mathbf{r},\mathbf{r}',\mathbf{r}_s) = G_0^*(\mathbf{r}_g,\mathbf{r}') G_0^*(\mathbf{r}',\mathbf{r}_s) G_0(\mathbf{r}_g,\mathbf{r}) G_0(\mathbf{r},\mathbf{r}_s), \tag{29}$$

the quantity

$$\begin{aligned} H_{cc}(\mathbf{r},\mathbf{r}') &= \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega \frac{\partial G^*(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_c(\mathbf{r}')} \frac{\partial G(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_c(\mathbf{r})} \\ &= \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega\omega^4 \left[1 + \beta s_{q_0}(\mathbf{r})\right]^2 \mathcal{G}(\mathbf{r}_g,\mathbf{r},\mathbf{r}',\mathbf{r}_s), \end{aligned} \tag{30}$$

coupling P-wave velocity variations with themselves, and

$$\begin{aligned} H_{cq}(\mathbf{r},\mathbf{r}') &= \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega \frac{\partial G^*(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_c(\mathbf{r}')} \frac{\partial G(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_q(\mathbf{r})} \\ &= \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega\omega^4 \left[1 + \beta s_{q_0}(\mathbf{r})\right] \left[\beta s_{q_0}(\mathbf{r})\right] \mathcal{G}(\mathbf{r}_g,\mathbf{r},\mathbf{r}',\mathbf{r}_s) \end{aligned} \tag{31}$$

and

$$\begin{aligned} H_{qc}(\mathbf{r},\mathbf{r}') &= \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega \frac{\partial G^*(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_q(\mathbf{r}')} \frac{\partial G(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_c(\mathbf{r})} \\ &= \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega \omega^4 \left[\beta s_{q_0}(\mathbf{r})\right] \left[1+\beta s_{q_0}(\mathbf{r})\right] \mathcal{G}(\mathbf{r}_g,\mathbf{r},\mathbf{r}',\mathbf{r}_s), \end{aligned} \tag{32}$$

which couple P-wave velocity with Q, and finally

$$\begin{aligned} H_{qq}(\mathbf{r},\mathbf{r}') &= \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega \frac{\partial G^*(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_q(\mathbf{r}')} \frac{\partial G(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_q(\mathbf{r})} \\ &== \sum_{\mathbf{r}_g,\mathbf{r}_s} \int d\omega \omega^4 \left[\beta s_{q_0}(\mathbf{r})\right]^2 \mathcal{G}(\mathbf{r}_g,\mathbf{r},\mathbf{r}',\mathbf{r}_s), \end{aligned} \tag{33}$$

which couples Q with itself, are all incorporated as follows. A Gauss-Newton update in the P-wave velocity and Q model parameters $[s_c, s_q]^T$ is computed by

$$\left[\begin{array}{c} \delta s_c(\mathbf{r}) \\ \delta s_q(\mathbf{r}) \end{array}\right] = \int d\mathbf{r}' \mathcal{H}^{-1}(\mathbf{r},\mathbf{r}') \int d\mathbf{r}'' \left[\begin{array}{cc} -H_{qq}(\mathbf{r}',\mathbf{r}'') & H_{cq}(\mathbf{r}',\mathbf{r}'') \\ H_{qc}(\mathbf{r}',\mathbf{r}'') & H_{cc}(\mathbf{r}',\mathbf{r}'') \end{array}\right] \left[\begin{array}{c} g_c(\mathbf{r}) \\ g_q(\mathbf{r}) \end{array}\right], \tag{34}$$

where the four H functions are given in equations (29)-(33) above, and the gradients are those given in equations (27)-(28). The generalized determinant is

$$\mathcal{H}(\mathbf{r},\mathbf{r}') = \int d\mathbf{r}'' \left[H_{cc}(\mathbf{r},\mathbf{r}'')H_{qq}(\mathbf{r}'',\mathbf{r}') - H_{cq}(\mathbf{r},\mathbf{r}'')H_{qc}(\mathbf{r}'',\mathbf{r}')\right], \tag{35}$$

and its reciprocal function is defined such that

$$\int d\mathbf{r}'' \mathcal{H}^{-1}(\mathbf{r},\mathbf{r}'')\mathcal{H}(\mathbf{r}'',\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}'). \tag{36}$$

QUESTION 1: SHOULD Q BE INCLUDED IN FWI?

With the mathematical framework for absorptive FWI in place, we can ask our questions. The first question is a little broad – a better way to put it is as follows: "Is it worthwhile incorporating Q in FWI when attenuation predominates at high frequencies and FWI focuses on low frequencies?" One answer to this is "Yes, because we would like to have FWI move into the higher frequencies too." But there is a more basic answer, which we are now in a position to give.

The answer lies in the an-acoustic sensitivities, i.e., equations (16)–(18), and (20)–(25). It turns out to hinge on attenuation and dispersion and their varied effect, so we will analyze the attenuation-only and dispersion-only cases. In fact, since the question has to do with whether or not we take the trouble to include Q, let us furthermore focus on the P-wave velocity sensitivities. We have

$$\frac{\partial G(\mathbf{r}_g,\mathbf{r}_s)}{\partial s_c(\mathbf{r})} \propto \omega^2 G_0^a(\mathbf{r}_g,\mathbf{r})G_0^a(\mathbf{r},\mathbf{r}_s), \tag{37}$$

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_c(\mathbf{r})} \propto \omega^2 G_0^d(\mathbf{r}_g, \mathbf{r}) G_0^d(\mathbf{r}, \mathbf{r}_s), \tag{38}$$

for the attenuation-only and dispersion-only cases respectively, and for reference we will also consider

$$\frac{\partial G(\mathbf{r}_g, \mathbf{r}_s)}{\partial s_c(\mathbf{r})} \propto \omega^2 G_0(\mathbf{r}_g, \mathbf{r}) G_0(\mathbf{r}, \mathbf{r}_s), \tag{39}$$

a completely acoustic sensitivity expression, achievable by taking the limit $Q \to \infty$ in any of the other examples. The proportionality constants are not as important in answering this question as the Green's functions are. Recalling that for every sensitivity we calculate a background model s_0 was introduced. It is in this medium that the Green's functions apply. The Green's functions satisfy equations (10–(11).

Let us plot the sensitivities in equations (37)–(39) for a few representative frequencies, assuming a homogeneous medium. In Figure 1 we focus on the damage attenuation can inflict on us if we wrongly assume the Earth is perfectly elastic. The left column are plots of the real parts of purely acoustic sensitivities, calculated as per equation (39), with c_0 =1500m/s, ranging from a low frequency value of 15Hz at the top, to a mid-range frequency value of 50Hz in the middle, and finally to a higher frequency value of 120Hz at the bottom. The middle column is equivalent in every way, except that the sensitivities are calculated using equation (37) with a finite $Q = 20$, and no dispersion. Here the expected trend is visible: at the low end (top row), the sensitivities are very similar to those of the purely acoustic case. As we move to higher frequencies, greater differences are visible. Thus, a preliminary conclusion: if a FWI inversion scheme is going to incorporate only low frequencies, it is not important to include Q.

However, the effect of dispersion has not been considered. A similar set of plots for the dispersion-only case is included in Figure 2. The left column is the same as in the previous case; the middle column now illustrates the sensitivities for Q =20, but using equation (38), i.e., including the dispersion component of wave propagation only. This time, the opposite trend is noticed, with the low and mid-range frequencies showing the largest discrepancies.

So, the answer to the question, when both attenuation and dispersion are properly accounted for, is a clear yes. Of course, we have not answered the question "how" in this paper, but a foray into that is made by Innanen (2015).

QUESTION 2: HOW DOES FWI ACCOMPLISH Q COMPENSATION?

The second question we can shed light on concerns the combined roles of the gradient and the inverse Hessian. The latter of the two has been discussed in many CREWES reports in recent years, and elsewhere, in terms of its mitigation of parameter cross-talk, suppression of artifacts caused by second order scattering processes, and its replication of the positive effects of using deconvolution rather than cross-correlation based imaging conditions in forming the gradient. This latter point is a useful clue for the problem of Q. The question is: how, in detail, does Q compensation happen in FWI?

First, we have to agree that Q compensation *does* happen in FWI if it is based on anelastic or an-acoustic wave physics. If we take an inclusive view of what Q compensa-

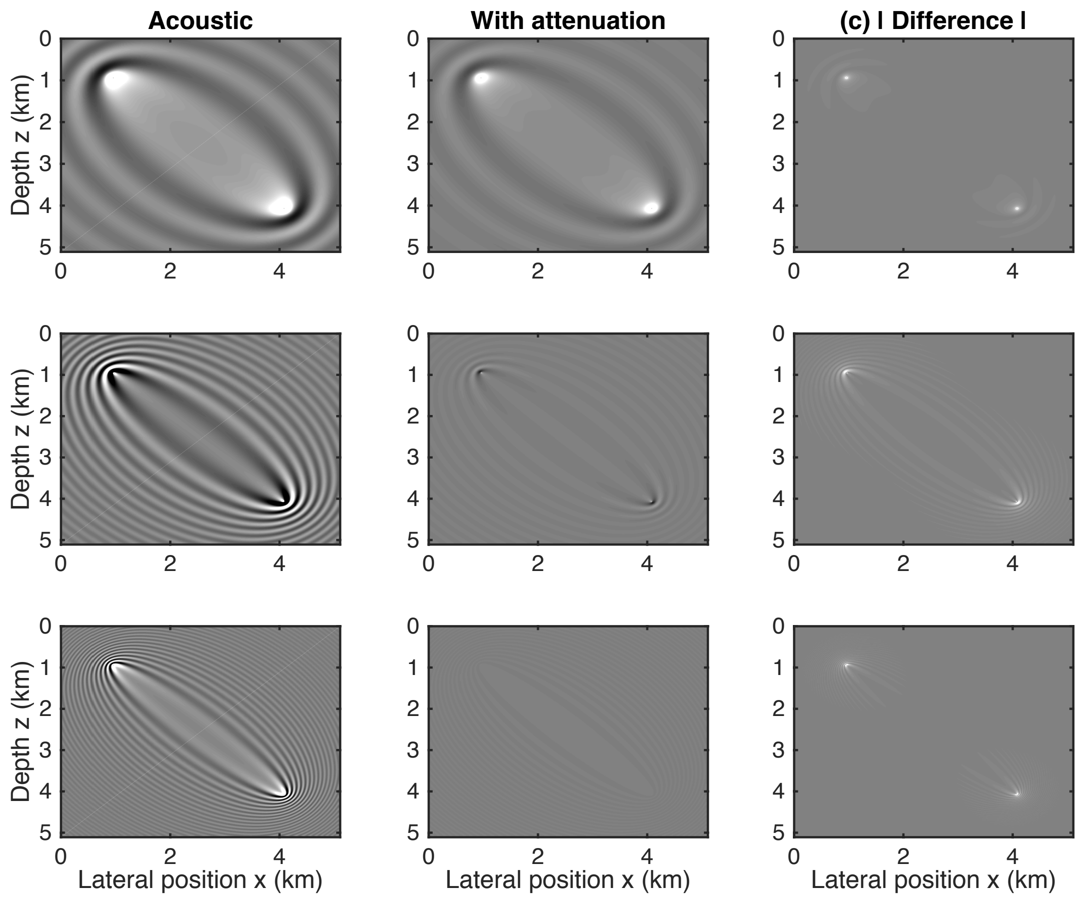

FIG. 1. Sensitivities for frequencies 15Hz (top row), 50Hz (middle row), and 120Hz (bottom row). The comparison is between purely acoustic sensitivities (left column), attenuation-only an-acoustic sensitivities (middle column), and their difference (right column).

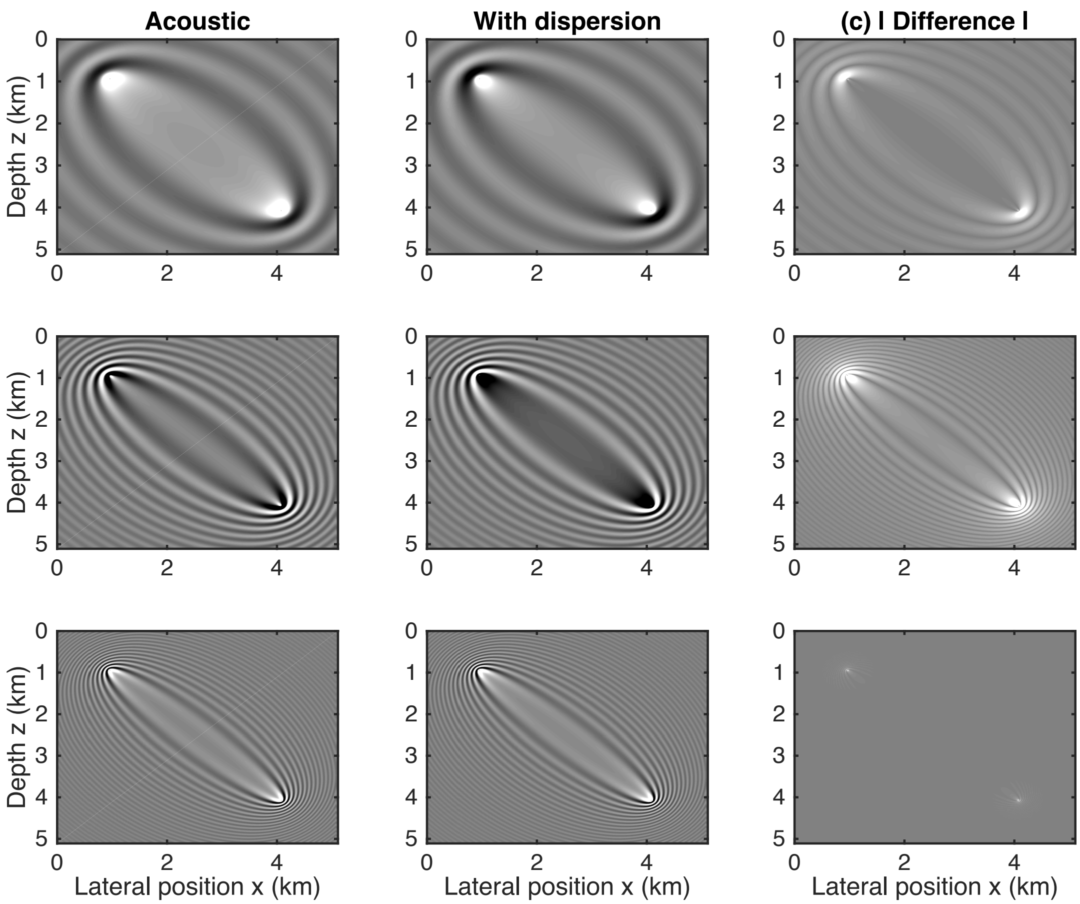

FIG. 2. Sensitivities for frequencies 15Hz (top row), 50Hz (middle row), and 120Hz (bottom row). The comparison is between purely acoustic sensitivities (left column), dispersion-only an-acoustic sensitivities (middle column), and their difference (right column).

tion means, then it certainly does. An inclusive viewpoint would hold that, since anelastic parameter profiles with sharp boundaries give rise to data with non-sharp (i.e., attenuated) events, the reconstruction of the former from the latter must involve some process of "boosting" high frequencies. To the extent we are willing to refer to this process as a flavour of Q compensation, Q compensation happens in any complete inversion scheme.

The core stage of FWI is the construction of the gradient; in gradient-based methods, the only additional step is to calculate a single scalar multiplier to complete an update. Let us first see that the an-acoustic gradient does not have the wherewithal to boost high frequencies attenuated during wave propagation. Consider the gradient g_q for a 1D, single parameter inversion scheme, with source and receiver coincident at $z_g = z_s = 0$. Equation (21) reduces to

$$\begin{aligned}\frac{\partial G(0,0)}{\partial s_q(z)} &= -\omega^2\left[\beta s_{q0}(z)\right]G_0(0,z)G_0(z,0)\\ &= -i\omega^2 s_{q0}\left[\frac{e^{iKz}}{i2K}\right]\left[\frac{e^{iKz}}{i2K}\right].\end{aligned} \tag{40}$$

If the background medium is attenuative, the propagation constant K has both real and imaginary components:

$$K = K_r + iK_i. \tag{41}$$

So, the sensitivity becomes

$$\frac{\partial G(0,0)}{\partial s_q(z)} = A(\omega)e^{i2K_r z}e^{-2K_i z}. \tag{42}$$

Now, in the formation of the gradient, the sensitivities are the kernel of an operator: $\int d\omega(\partial G/\partial s)[\cdot]$, which acts on the complex conjugate of the residuals:

$$g_q(z) = -\int d\omega A(\omega)e^{i2K_r z}e^{-2K_i z}\delta P^*(\omega). \tag{43}$$

Now, this construction does *not* boost the high frequencies attenuated in the residuals. In fact, it does the opposite. The residuals at high frequencies are *further attenuated* by the $e^{-2K_i z}$ of the operator. So, we must not seek in gradient based methods a model reconstruction which reintroduces the small wavelengths properly; the reconstruction will have been doubly attenuated.

Is this compensated for in the inverse Hessian. In a 1 parameter problem, H_{qq} from equation (33) is the complete Hessian. In 1D, it reduces to

$$\begin{aligned}H(z,z') &= \int d\omega \frac{\partial G(0,0)}{\partial s_q(z)}\frac{\partial G^*(0,0)}{\partial s_q(z')}\\ &= \int d\omega\omega^4 s_{q0}^2\left[\frac{e^{i2Kz}}{i2K}\right]\left[\frac{e^{-i2K^*z'}}{-i2K^*}\right].\end{aligned} \tag{44}$$

Once again breaking K up into real and imaginary parts, the Hessian takes on the form:

$$H(z, z') = \int d\omega B(\omega) e^{i2K_r(z-z')} e^{-2K_i(z+z')}. \tag{45}$$

In acoustic settings, $K_i \to 0$, and B goes over such that H becomes a weighted delta function. With finite K_i, the right hand term attenuates proportionally to the sum of z and z'. Inverting this operator, the effect is to correct for attenuation, boosting proportionally to this sum. So, it is the residual-independent component of the Hessian which corrects the small model wavelengths for Q compensation as the model is constructed.

CONCLUSIONS

Two questions concerning the inclusion in seismic FWI of Q_P are considered. The first question is whether or not to do so: anecdotally, attenuation parameters have been suggested to be unimportant to FWI, since they dominate at high frequencies and FWI is primarily concerned with low. Because of the accompanying phenomenon of dispersion, the answer is no. The second is: where in the mechanisms of FWI (e.g., within the machinery of the gradient and the various approximate Hessians), are the tasks we normally associate with Q contained? Is there a component of the Hessian operator tasked with Q compensation, for instance. Indeed, in fact, the gradient alone aggravates attenuation beyond what is nominally present in the residuals.

ACKNOWLEDGEMENTS

We thank the sponsors of CREWES for continued support. This work was funded by CREWES industrial sponsors and NSERC (Natural Science and Engineering Research Council of Canada) through the grant CRDPJ 461179-13.

REFERENCES

Aki, K., and Richards, P. G., 2002, Quantitative Seismology: University Science Books, 2nd edn.

Innanen, K. A., 2015, Direct measurement of near surface seismic body wave dispersion from uncorrelated vibroseis data: CREWES Annual Report, **27**.