Viscoelastic AVO equations: Born versus Aki-Richards approximations

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ABSTRACT

Anelastic properties of reservoir rocks are important and sensitive indicators of fluid saturation and viscosity changes due (for instance) to steam injection. The description of seismic waves propagating through viscoelastic continua is quite complex, involving a range of unique homogeneous and inhomogeneous modes. This is true even in the relatively simple theoretical environment of amplitude-variation-with-offset (AVO) analysis. For instance, a complete treatment of the problem of linearizing the solutions of the low-loss viscoelastic Zoeppritz equations, to obtain an extended Aki-Richards approximation (one that is in accord with the appropriate complex Snell's law) is lacking in the literature. Also missing is a clear analytical path allowing such forms to be reconciled with more general volume scattering pictures of viscoelastic seismic wave propagation. Our analysis, which provides these two missing elements, leads to approximate reflection and transmission coefficients for the P- and types I and II S waves as formulated by Borcherdt. These involve additional, complex, terms alongside those of the standard isotropic-elastic Aki-Richards equation. The extra terms are shown to have a significant influence on reflection strengths, particularly when the degree of inhomogeneity is high. The particular AVO forms we present are finally shown to be special cases of potentials for volume scattering from viscoelastic inclusions.

INTRODUCTION

Recently a volume scattering picture of viscoelastic seismic waves has been developed for the purposes of modeling, processing and inversion of seismic data exhibiting non-negligible intrinsic attenuation (Moradi and Innanen, 2015). That work, which culminates in the derivation of the mathematical form of the viscoelastic scattering potential, can be understood as the extension of the exact layered-medium results of (Borcherdt, 2009) to a linearized but fully multidimensional framework. It adds, to the toolbox for the quantitative analysis of homogeneous and inhomogeneous anelastic waves, a perturbation-based approach, to sit alongside complex ray-based techniques (Hearn and Krebes, 1990) and numerical techniques (Carcione et al., 1988a,b; Carcione, 1993; Robertsson et al., 1994; Carcione, 2001).

From the point of view of practical exploration and monitoring geophysics, the consequences of the scattering result are twofold. First, all direct inverse scattering target identification/inversion methods are formulated beginning with the framing of an appropriate scattering potential (Weglein et al., 2003, 2009). So, the new framework permits a range of viscoacoustic inverse scattering results (Innanen and Weglein, 2007; Innanen and Lira, 2010) now to be posed for the more complete attenuating elastic case. Second, the scattering potential is also a useful starting point in the construction of Frechet kernels for full waveform inversion (Fichtner, 2010; Fichtner and van Driel, 2014). If it is desirable to include some particular observable viscoelastic phenomenon (e.g., inhomogeneous wave modes) in a full waveform inversion procedure, the Frechet kernel must be general enough to admit that phenomenon. So, the new viscoelastic result also makes possible the derivation of general full waveform inversion formulas for attenuating media.

There remain several outstanding questions regarding the relationship between the newer viscoelastic volume scattering picture and the older stratified medium picture. The purpose of this paper is to address those questions.

In exploration and monitoring geophysics, backscattered seismic amplitudes from stratified media fit into processing flows through AVO/AVA technology (Castagna and Backus, 1993; Foster et al., 2010). The workhorse formula within this technology is the Aki-Richards approximation (Aki and Richards, 2002), wherein exact displacement reflection coefficients RPP, RPS, RSP and RSS are linearized with respect to perturbations in elastic properties across a reflecting boundary. Stolt and Weglein Stolt and Weglein (2012) have shown that there is a close relationship between the isotropic-elastic scattering potential and the Aki-Richards approximation, the former reducing to the latter for small contrasts and small opening angles. It follows that a similar reduction of the viscoelastic case should lead to formulas corresponding to a viscoelastic-type Aki-Richards approximation. A confirmation of this expectation, and the detailed process by which it occurs, are outstanding issues.

Anelastic reflection coefficients have been discussed analytically (White, 1965; Krebes, 1984; Ursin and Stovas, 2002; Zhao et al., 2014) and numerically (Samec and Blangy, 1992), and in the context of a variety of linear approximations, both in isotropic and anisotropic settings (Behura and Tsvankin, 2006a,b; Innanen, 2011). These latter formulas are examples of anelastic Aki-Richards approximation, and so they belong to the same class of formulas in which we expect the reduced version of the general viscoelastic scattering potential to belong. Formulas of this kind can be used to drive anelastic inversion procedures, both linear and nonlinear (Innanen, 2011); or, alternatively, via examination of the frequency rate of change of reflection coefficients (Innanen, 2012). Techniques of this kind become increasingly relevant as evidence accrues that anelastic amplitude signatures provide direct information about reservoir fluids (Ostrander, 1984; Chapman et al., 2006; Odebeatu et al., 2006; Schmalholz and Podladchikov, 2009; Ren et al., 2009; Wu et al., 2014)

The general process of an inhomogeneous viscoelastic plane wave interacting with a planar horizontal boundary is quite complicated. Thus far no linearization of the exact equations for this reflection and transmission problem has been presented in the literature wherein general inhomogeneity is accommodated. A key result in this paper is the provision of such a linearization (i.e., a viscoelastic Aki-Richards approximation), and the demonstration that the viscoelastic volume scattering model reduces to it.

A full linearization procedure must take into account in detail both specialized anelastic Zoeppritz equations and the complex ray parameter/vertical slowness vector as input to those equations. We begin with the latter wave quantities, determining the relationship between perturbations in elastic P- and S-wave velocities and quality factors across a reflecting boundary and the resulting perturbations in the P- and S-wave attenuation angles (i.e., the angles between planes of constant phase and planes of constant amplitude). We then write down the Zoeppritz equations, formulated for reflection, transmission and conversion of plane anelastic P, type-I S and type-II S waves (see Borcherdt, 2009 for a complete discussion of P, SI and SII modes). These lead to exact, though rather complicated, expressions for all requisite reflection and transmission coefficients. Next, the P-P, P-SI, and SI-SI coefficients are linearized by considering the effect of weak contrasts on both the complex Snell's law and the Zoeppritz equations. Finally we demonstrate the consistency between the linearized viscoelastic reflection coefficient expressions and the viscoelastic scattering potentials as derived in the general volume scattering development of (Moradi and Innanen, 2015).

The paper is organized as follows. In section 2, we briefly introduce the viscoelastic waves and define the ray parameter, slowness and polarization vectors for a low-loss viscoelastic medium. In section 3, we study the Snell's law and its linearized form in a viscoelastic medium. In section 4, we obtain the exact reflectivities and linearized them in section 5. In section 6, we obtain the relationship between the linearized reflectivity and the scattering potential. Finally, we summarize our results and suggest possible directions for further research in section 7.

VISCOELASTIC RAY PARAMETERS AND SLOWNESSES

Linearized AVO analysis requires the definition of polarization and slowness vectors. In a viscoelastic medium, the wavenumber vector is a complex vector whose real part characterizes the direction of wave propagation and imaginary part characterizes the attenuation of the wave. The direction of maximum attenuation of a plane wave can differ from the propagation direction. The wavenumber vector of inhomogeneous waves is represented by

$$\mathbf{K} = \mathbf{P} - i\mathbf{A}. \tag{1}$$

Here $\mathbf{P}$ is the propagation vector perpendicular to the constant phase plane $\mathbf{P}\cdot\mathbf{r} = constant$, and $\mathbf{A}$ is the attenuation vector perpendicular to the amplitude constant plane $\mathbf{A}\cdot\mathbf{r} = constant$. The attenuation vector $\mathbf{A}$ is in the direction of maximum decrease of amplitude. In the case that attenuation and propagation vectors are in the same direction, the wave is said to be homogeneous. An elastic media is represented by $\mathbf{A} = 0$. If we represent the angle between $\mathbf{P}$ and $\mathbf{A}$ by δ, for inhomogeneous waves $0 < \delta < \pi/2$. Thus in the presence of attenuation, the wavenumber vector is complex, with its real part displaying propagation direction and its imaginary part refering to the direction of maximum wave attenuation. As a consequence, P- and S-velocities in viscoelastic media are generalized to the elastic P- and S-velocities V_{PE} and V_{SE}:

$$V_{\mathrm{P}} = V_{\mathrm{PE}}\left(1 + i\frac{Q_{\mathrm{P}}^{-1}}{2}\right), \tag{2}$$

$$V_{S} = V_{\mathrm{SE}}\left(1 + i\frac{Q_{\mathrm{S}}^{-1}}{2}\right), \tag{3}$$

where Q_{P} and Q_{S} are the quality factors for P- and S waves respectively. The displacement vectors for P- and SI-waves are (Borcherdt, 2009)

$$\mathbf{U}_{\mathrm{P}} = \boldsymbol{\xi}_{\mathrm{P}}\Phi_0 \exp\left[-i(\mathbf{K}_{\mathrm{P}}\cdot\mathbf{r} - \omega t)\right], \tag{4}$$

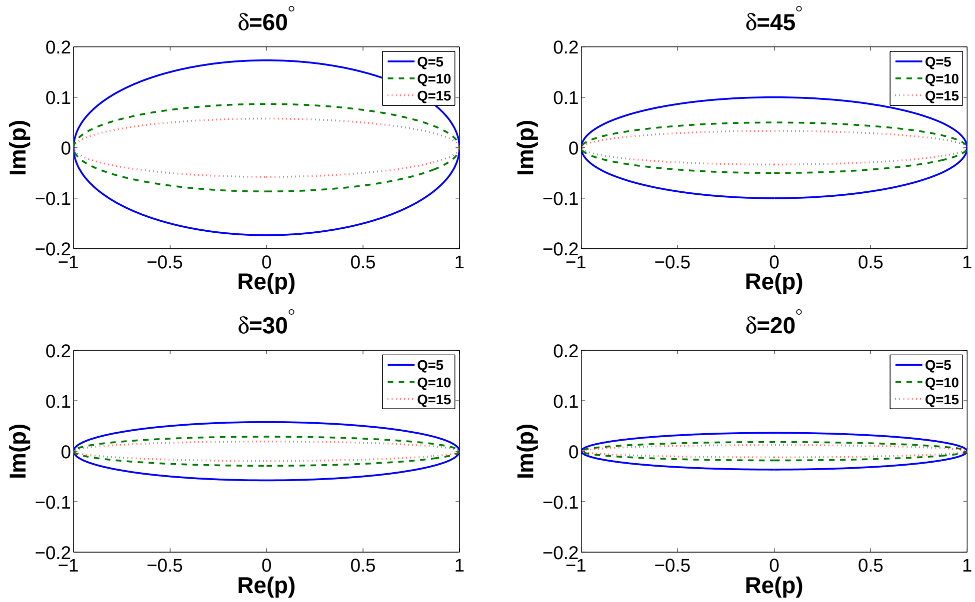

FIG. 1. Diagram illustrating the complex ray parameter for various values of reciprocal quality factor Q and attenuation angle δ.

$$\mathbf{U}_S = \boldsymbol{\zeta}_S \Psi_0 \exp\left[-i(\mathbf{K}_S \cdot \mathbf{r} - \omega t)\right], \tag{5}$$

where Φ_0 and Ψ_0 are complex scalar constants and $\boldsymbol{\xi}_{\mathrm{P}}$ and $\boldsymbol{\zeta}_S$ are respectively the polarization vectors for P- and SI-waves

$$\boldsymbol{\xi}_{\mathrm{P}} = \frac{1}{\omega} V_{\mathrm{P}} \mathbf{K}_{\mathrm{P}} = \frac{V_{\mathrm{PE}}}{\omega} \left\{ \mathbf{K}_{\mathrm{P}} + \frac{i}{2} Q_{\mathrm{P}}^{-1} \mathbf{P}_{\mathrm{P}} \right\}, \tag{6}$$

$$\boldsymbol{\zeta}_S = \frac{1}{\omega} V_S \mathbf{K}_S = \frac{V_{\mathrm{SE}}}{\omega} \left\{ \mathbf{K}_S + \frac{i}{2} Q_{\mathrm{S}}^{-1} \mathbf{P}_S \right\} \times \mathbf{n}, \tag{7}$$

where $\mathbf{n}$ is a unit vector orthogonal to the plane formed by $\mathbf{P}_{\mathrm{S}}$ and $\mathbf{A}_{\mathrm{S}}$. Particle motion related to the displacement for P is an ellipse.

The above results apply for viscoelastic plane waves propagating in an isotropic homogeneous medium. What happens if an inhomogeneous wave with a elliptical polarization hits the boundary between two half-spaces? To answer this question we define two half-spaces with different physical properties separated by a planar boundary. The analysis of the Zoeppritz equation and continuity of displacements and stresses across the boundary are similar to the elastic case. The difference is that ray parameters and vertical slownesses are complex. For example let us consider the incident P-wave

$$\mathbf{U}_{\mathrm{P}}^{\downarrow} \simeq (\xi_x \mathbf{x} + \xi_z \mathbf{z}) \exp\left\{i\omega(px + q_{\mathrm{P}} z)\right\}, \tag{8}$$

where $\downarrow$ indicates the direction of the vertical component of the incident wave propagation vector, and where the complex ray parameter p and vertical slowness q_{P} are defined as

$$p = \frac{1}{V_{\mathrm{PE}}} \left[\sin\theta_{\mathrm{P}} \left(1 - i \frac{Q_{\mathrm{P}}^{-1}}{2} \right) + \frac{i}{2} Q_{\mathrm{P}}^{-1} \cos\theta_{\mathrm{P}} \tan\delta_{\mathrm{P}} \right], \tag{9}$$

$$q_{\mathrm{P}} = \frac{1}{V_{\mathrm{PE}}}\left[\cos\theta_{\mathrm{P}}\left(1 - i\frac{Q_{\mathrm{P}}^{-1}}{2}\right) - \frac{i}{2}Q_{\mathrm{P}}^{-1}\sin\theta_{\mathrm{P}}\tan\delta_{\mathrm{P}}\right]. \tag{10}$$

In addition we define the x- and z-components of the polarization vectors as

$$\xi_x = pV_{\mathrm{P}} = \sin\theta_{\mathrm{P}} + \frac{i}{2}Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}}\cos\theta_{\mathrm{P}}, \tag{11}$$

$$\xi_z = q_{\mathrm{P}}V_{\mathrm{P}} = \cos\theta_{\mathrm{P}} - \frac{i}{2}Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}}\sin\theta_{\mathrm{P}}. \tag{12}$$

Analogous expressions hold for the case of an incident S-wave. We confirm our results by observing that the complex ray parameter, vertical slowness and polarization components satisfy the following relations

$$p^2 + q_{\mathrm{P}}^2 = \frac{1}{V_{\mathrm{P}}^2} = \frac{1}{V_{\mathrm{PE}}^2}(1 + iQ_{\mathrm{P}}^{-1}), \tag{13}$$

$$\xi_x^2 + \xi_z^2 = 1. \tag{14}$$

In Figure 1, we plot the complex ray parameter versus phase and attenuation angles for various values of quality factor Q in the complex plane. It can be seen that the ray parameter for a viscoelastic medium is an ellipse whose eccentricity grows for smaller values of attenuation angle.

VISCOELASTIC SNELL'S LAW IN THE LOW-CONTRAST APPROXIMATION

Consider two homogeneous viscoelastic half spaces separated by a plane interface. All properties and quantities related to the upper and lower half spaces are labeled respectively by subscripts 1 and 2. Snell's law expresses the relationship between incident and transmitted angles and velocities before and after the reflection or transmission of waves. The study of Snell's law is required for several reasons: first among them is that we can analyse the homogeneity or inhomogeneity of the reflected and transmitted waves by having the homogeneity of incident wave. Secondly, in the process of linearization we need to obtain the perturbation in phase and attenuation angles in terms of perturbations in physical properties. Snell's law for viscoelastic materials is discussed by Wennerberg Wennerberg (1985) and Borcherdt Borcherdt (2009). Since the ray parameter in a viscoelastic medium is complex, the generalized snell's law has two parts, real and imaginary.

Snell's law is based on the fact that the horizontal slowness (ray parameter) is conserved during the reflection and transmission from a boundary. For a viscoelastic medium, the ray parameter not only depends on the phase angle but also on the attenuation angle. Also it is a complex quantity whose real part is the elastic ray parameter given by

$$p_E = \frac{\sin\theta_{P1}}{V_{PE1}} = \frac{\sin\theta_{P2}}{V_{PE2}} = \frac{\sin\theta_{S1}}{V_{SE1}} = \frac{\sin\theta_{S2}}{V_{SE2}}, \tag{15}$$

and the imaginary part

$$p_A = \frac{Q_{P1}^{-1}}{2}(p_E - q_{PE1}\tan\delta_{P1}) = \frac{Q_{P2}^{-1}}{2}(p_E - q_{PE2}\tan\delta_{P2}), \tag{16}$$

$$= \frac{Q_{S1}^{-1}}{2}(p_E - q_{SE1}\tan\delta_{S1}) = \frac{Q_{S2}^{-1}}{2}(p_E - q_{SE2}\tan\delta_{S2}), \tag{17}$$

where

$$q_{SE} = \frac{\cos\theta_S}{V_{SE}}, \qquad q_{PE} = \frac{\cos\theta_P}{V_{PE}}, \tag{18}$$

are the elastic vertical slownesses. Here θ_{P1} is the angle for incident P-wave, θ_{P2} the angle for transmitted P-wave, θ_{S1} the angle of reflected S-wave and θ_{S2} angle of transmitted S-wave. Using the imaginary part of Snell's law we can analyze the conditions for homogeneity and inhomogeneity of the reflected and transmitted waves (Borcherdt, 2009, 1982). For example, for an incident P-wave, the angle of incidence is equal to the angle of the reflected P-wave, which is confirmed using the real part of Snell's law. In this case the imaginary part of Snell's law ensures that the attenuation angle for incident and reflected waves are equal. As a result the reflected P-wave is homogeneous if and only if the incident wave is homogeneous. Let us consider the case where we have an incident inhomogeneous P-wave. In this case the transmitted attenuation angle for the P-wave, in terms of attenuation angle and incident angle is given by

$$\tan\delta_{P2} = \left(\frac{V_{PE2}}{V_{PE1}}\right)\frac{\sin\theta_{P1} - \frac{Q_{P2}}{Q_{P1}}\left[\sin\theta_{P1} - \cos\theta_{P1}\tan\delta_{P1}\right]}{\sqrt{1-\left(\frac{V_{PE2}}{V_{PE1}}\right)^2\sin^2\theta_{P1}}}. \tag{19}$$

For reflected and transmitted S-wave, respectively we have

$$\tan\delta_{S1} = \left(\frac{V_{SE1}}{V_{PE1}}\right)\frac{\sin\theta_{P1} - \frac{Q_{S1}}{Q_{P1}}\left[\sin\theta_{P1} - \cos\theta_{P1}\tan\delta_{P1}\right]}{\sqrt{1-\left(\frac{V_{SE1}}{V_{PE1}}\right)^2\sin^2\theta_{P1}}}, \tag{20}$$

$$\tan\delta_{S2} = \left(\frac{V_{SE2}}{V_{PE1}}\right)\frac{\sin\theta_{P1} - \frac{Q_{S2}}{Q_{P1}}\left[\sin\theta_{P1} - \cos\theta_{P1}\tan\delta_{P1}\right]}{\sqrt{1-\left(\frac{V_{SE2}}{V_{PE1}}\right)^2\sin^2\theta_{P1}}}. \tag{21}$$

For an incidence P-wave, if $V_{PE2} > V_{SE2} > V_{PE1}$, when $\theta_{P1} \to 90^\circ$, the wave is refracted rather than transmitted. In this case Snell's law predicts two critical angles, one for the refracted P-wave and the other for the refracted S-wave.

Let us consider the special case in which there is no contrast in P-wave quality factors $Q_{P1} = Q_{P2}$. In this case

$$\tan\delta_{P2} = \frac{\left(\frac{V_{PE2}}{V_{PE1}}\right)\cos\theta_{P1}}{\sqrt{1-\left(\frac{V_{PE2}}{V_{PE1}}\right)^2\sin^2\theta_{P1}}}\tan\delta_{P1}. \tag{22}$$

This equation shows that even if there is no contrast in the P-wave quality factor, the reflected and transmitted attenuation angles are different. In other words, a P-wave contrast alone can cause a change in the attenuation angle. At normal incidence, $\theta_{P1} = 0$, we have

$$\tan\delta_{P2} = \left(\frac{V_{PE2}Q_{P2}}{V_{PE1}Q_{P1}}\right)\tan\delta_{P1}, \tag{23}$$

$$\tan\delta_{S1} = \left(\frac{V_{SE1}Q_{S1}}{V_{PE1}Q_{P1}}\right)\tan\delta_{P1}, \tag{24}$$

$$\tan\delta_{S2} = \left(\frac{V_{SE2}Q_{S2}}{V_{PE1}Q_{P1}}\right)\tan\delta_{P1}. \tag{25}$$

So at normal incidence, if the incident P-wave is an (in)homogeneous wave, the reflected or transmitted S- and P-wave are (in)homogenous.

To study the contributions of the jumps in elastic and anelastic properties to the reflectivities, we must linearize the reflection amplitudes. To calculate the approximate reflectivities for a low contrast model, and to write the physical quantities in medium 1, and medium 2 in terms of fractional perturbations, we must express the phase and attenuation angles in perturbed form. This in turn requires us to linearize the generalized Snell's law.

The main assumption required to extract the approximate form of the reflectivities is that the physical properties in the two layer are only slightly different. Other words, we can define fractional changes in properties as perturbations which are much smaller than one. This procedure is straightforward for properties of the medium like density, velocities and quality factors. For example the density of the layer above the reflector is given by

$$\rho_1 = \bar{\rho}\left(1 - \frac{1}{2}\frac{\Delta\rho}{\bar{\rho}}\right), \tag{26}$$

while the density in the layer below the reflector is

$$\rho_2 = \bar{\rho}\left(1 + \frac{1}{2}\frac{\Delta\rho}{\bar{\rho}}\right). \tag{27}$$

Similar expressions are valid for V_{P}, V_S, Q_{P} and Q_{S}. In the above relations Δ, refers to the difference in the lower and upper layers and bar indicates the average of the quantities. In the final form of the linearized reflectivity we shouldn't have any quantities related explicitly to either the upper or lower medium. Hence we need to express the phase and attenuation angles in terms of corresponding perturbations. This can be done by linearization of Snell's law. In the previous section we saw that Snell's law has both real and imaginary parts. By applying the linearization to the real part we obtain the perturbation in phase angle in terms of the perturbation in the corresponding velocity, weighted by the average of the phase angle. Using the linearization of the imaginary part, we obtain that perturbation in the attenuation angle in terms of the perturbations in the corresponding velocities and quality factors. In what follows any quantity related to the material property, slowness vector or angles without subscripts 1 or 2, stands for average of that quantity. The real part of Snell's law for P-wave results

$$\frac{\sin\theta_{P1}}{V_{P1}} = \frac{\sin\theta_{P2}}{V_{P2}}. \tag{28}$$

Using the expressions (26) and (27) for incidence phase angle for P-wave, θ_{P1}, and transmitted phase angle θ_{P2}, we expand the sin functions as

$$\sin\theta_{P1} = \sin\theta_{\mathrm{P}}\left(1 - \frac{1}{2}\frac{\Delta\theta_{\mathrm{P}}}{\tan\theta_{\mathrm{P}}}\right), \tag{29}$$

$$\sin\theta_{P2} = \sin\theta_{\mathrm{P}}\left(1 + \frac{1}{2}\frac{\Delta\theta_{\mathrm{P}}}{\tan\theta_{\mathrm{P}}}\right). \tag{30}$$

Inserting (29) and (30) and corresponding expressions for V_{P1} and V_{P2} in terms of average and differences in the P-wave velocity, we obtain the difference in the incidence and transmitted angle in terms of fractional perturbation in the P-wave velocity

$$\Delta\theta_{\mathrm{P}} \approx \frac{\Delta V_{\mathrm{P}}}{V_{\mathrm{P}}}\tan\theta_{\mathrm{P}}. \tag{31}$$

A similar expression holds for the θ_S. To obtain the linearized form of the reflectivities, we need to write the all quantities in terms of the fractional perturbations. Now, consider the imaginary part of the Snell's law

$$\frac{Q_{P2}}{Q_{P1}}\frac{V_{PE2}}{V_{PE1}}\frac{\cos\delta_{P2}}{\cos\delta_{P1}} = \frac{\sin(\theta_{P2} - \delta_{P2})}{\sin(\theta_{P1} - \delta_{P1})} \tag{32}$$

By expansion of the cosine functions in terms of differences and averages in attenuation angle we arrive at

$$\cos\delta_{P1} = \cos\delta_{\mathrm{P}}\left(1 + \frac{1}{2}\tan\delta_{\mathrm{P}}\Delta\delta_{\mathrm{P}}\right), \tag{33}$$

$$\cos\delta_{P2} = \cos\delta_{\mathrm{P}}\left(1 - \frac{1}{2}\tan\delta_{\mathrm{P}}\Delta\delta_{\mathrm{P}}\right). \tag{34}$$

Using the equations (29) and (30) for sin of δ_{P1}, δ_{P2}, θ_{P1}, θ_{P2} and the corresponding relation for velocities and quality factors in terms of perturbations we arrive at

$$\Delta\delta_{\mathrm{P}} = \frac{1}{2}\sin 2\delta_{\mathrm{P}}\left\{\frac{\Delta V_{\mathrm{PE}}}{V_{\mathrm{PE}}}\frac{1}{\cos^2\theta_{\mathrm{P}}} + \left(1 - \frac{\tan\theta_{\mathrm{P}}}{\tan\delta_{\mathrm{P}}}\right)\frac{\Delta Q_{\mathrm{P}}}{Q_{\mathrm{P}}}\right\}, \tag{35}$$

and similarly for the S-wave

$$\Delta\delta_S = \frac{1}{2}\sin 2\delta_S\left\{\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}\frac{1}{\cos^2\theta_S} + \left(1 - \frac{\tan\theta_S}{\tan\delta_S}\right)\frac{\Delta Q_{\mathrm{S}}}{Q_{\mathrm{S}}}\right\}. \tag{36}$$

As a result, perturbation in attenuation angle can be expressed in terms of perturbation in elastic velocities and quality factors. Also perturbation in attenuation angle depends to the average angle θ.

EXACT REFLECTION/TRANSSMISSION COEFFICIENTS

It has been shown that waves with elliptical polarization can not be converted to waves with the linear polarizations (Borcherdt, 2009; Moradi and Innanen, 2015). For example SII wave that has a linear polarization does not convert to P or SI waves. As a result we can write the reflection transmission for P- and SI-waves as a 4×4 matrix given by

$$\mathrm{R} = \begin{pmatrix} {}^{\downarrow}\mathrm{PP}^{\uparrow} & {}^{\downarrow}\mathrm{SIP}^{\uparrow} & {}^{\uparrow}\mathrm{PP}^{\uparrow} & {}^{\downarrow}\mathrm{SIP}^{\uparrow} \\ {}^{\downarrow}\mathrm{PSI}^{\uparrow} & {}^{\downarrow}\mathrm{SISI}^{\uparrow} & {}^{\downarrow}\mathrm{SISI}^{\uparrow} & {}^{\uparrow}\mathrm{SISI}^{\uparrow} \\ {}^{\downarrow}\mathrm{PP}^{\downarrow} & {}^{\downarrow}\mathrm{SIP}^{\downarrow} & {}^{\uparrow}\mathrm{PP}^{\downarrow} & {}^{\uparrow}\mathrm{SIP}^{\downarrow} \\ {}^{\downarrow}\mathrm{PSI}^{\downarrow} & {}^{\downarrow}\mathrm{SISI}^{\downarrow} & {}^{\uparrow}\mathrm{PSI}^{\downarrow} & {}^{\uparrow}\mathrm{SISI}^{\downarrow} \end{pmatrix}. \tag{37}$$

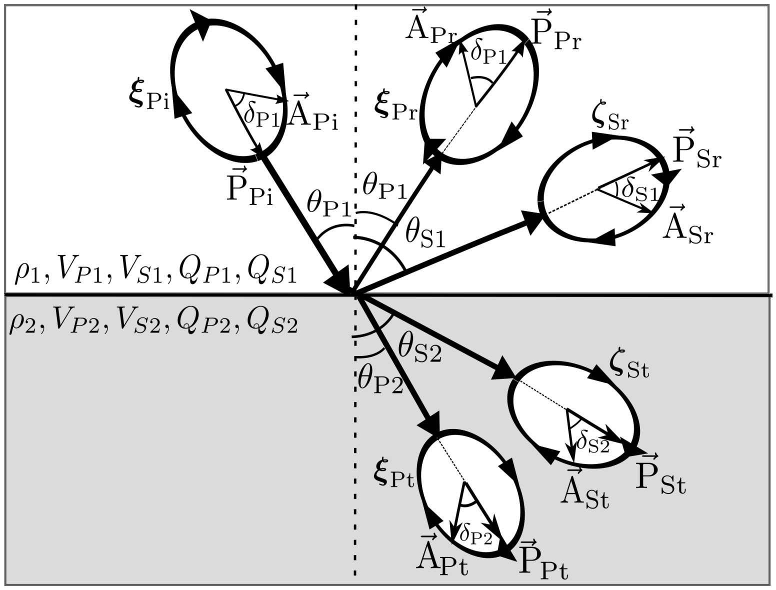

FIG. 2. Schematic diagram showing definitions of the phase and attenuation angles of the incident, reflected, and transmitted rays of an incident P-wave with non-normal incidence. Medium 1 is defined by its P-wave velocity $V_{\rm P1}$, S-wave velocity $V_{\rm S1}$, P-wave quality factor $Q_{\rm P1}$, S-wave quality factor $Q_{\rm S1}$ and its density ρ_1; and for medium 2, by $V_{\rm P2}, V_{\rm S2}, Q_{\rm P2}, Q_{\rm S2}$ and ρ_2. Angles are defined as, $\theta_{\rm P1}$ for the incident and reflected P-wave in medium 1, $\theta_{\rm S1}$ for the reflected SI-wave, and $\theta_{\rm P2}$ and $\theta_{\rm S2}$ respectively for transmitted P- and SI-waves. Attenuation angle for incident and reflected P-wave is given by $\delta_{\rm P1}$, also attenuation angles for reflected SI-wave, transmitted P- and SI-waves respectively are given by $\delta_{\rm S1}$, $\delta_{\rm S2}$ and $\delta_{\rm P2}$. $\xi_{\rm Pi}$, $\xi_{\rm Pr}$ and $\xi_{\rm Sr}$ respectively denotes the complex polarization vectors for incident, reflected and transmitted P-waves and $\xi_{\rm Pt}$ and $\xi_{\rm St}$ are the polarizations for the transmitted P- and SI-waves respectively.

The first letter refers the type of incident wave, and the second letter denotes the type of reflected or transmitted wave. The downward arrow $\downarrow$ indicates a wave traveling downward and $\uparrow$ indicates a wave traveling upward. So that a combination $\downarrow\uparrow$ refers a reflection coefficient, and a combination $\uparrow\uparrow$ indicates a transmission coefficient. The diagonal elements of the reflection-transmission matrix represents the reflections that preserve the type of the waves. For example $^{\downarrow}\mathrm{PP}^{\uparrow}$ refers to the reflected upgoing P-wave from downgoing incidence P-wave and similar explanations for other diagonal elements. On the other hand some off-diagonal elements indicate converted waves. For instance, $^{\downarrow}\mathrm{SIP}^{\uparrow}$ denotes a reflected upgoing P from incidence downgoing SI wave. Other off-diagonal elements refer to transmitted waves either converted modes or preserved modes. For example $^{\downarrow}\mathrm{SISI}^{\downarrow}$ is related to the transmitted downgoing SI wave from a downgoing incidence SI wave and $^{\downarrow}\mathrm{SIP}^{\downarrow}$ is a downgoing transmitted P wave from a downgoing incidence SI wave. For non-normal incidence, an incident P-wave generates reflected P- and S-waves and transmitted P- and SI-waves. The reflection and transmission coefficients depend on the angle of incidence and attenuation as well as on the material properties of the two layers. Fig. 2, is a schematic description of the reflection/transmission problem for an incident inhomogeneous P-wave. The displacements for incident, reflected and transmitted waves are Here the angle of incidence and reflection for the P-wave is defined by $\theta_{\rm P1}$, the angle of the reflected SI-wave defined by $\theta_{\rm S1}$, and $\theta_{\rm P2}$, and $\theta_{\rm S2}$ are the angles respectively for transmitted P- and SI waves. After solving the Zoeprittz equations, the reflection coefficients are given by (Ikelle and Amundsen, 2005)

$$(^{\downarrow}\mathrm{PP}^{\uparrow}) = \frac{c_1 d_2 - c_3 d_4}{d_1 d_2 + d_3 d_4}, \tag{38}$$

$$(^{\downarrow}\mathrm{PSI}^{\uparrow}) = -\left(\frac{V_{P1}}{V_{S1}}\right) \frac{c_3 d_1 + c_1 d_3}{d_1 d_2 + d_3 d_4}, \tag{39}$$

$$(^{\downarrow}\mathrm{SISI}^{\uparrow}) = -\frac{c_2 d_1 + c_4 d_3}{d_1 d_2 + d_3 d_4}, \tag{40}$$

where

$$d_1 = -2p^2 \Delta M (q_{P_1} - q_{P_2}) + (\rho_1 q_{P_2} + \rho_2 q_{P_1}), \tag{41}$$

$$d_2 = -2p^2 \Delta M (q_{S_1} - q_{S_2}) + (\rho_1 q_{S_2} + \rho_2 q_{S_1}), \tag{42}$$

$$d_3 = -p\left[2\Delta M (q_{P_1} q_{S_2} + p^2) - \Delta\rho\right], \tag{43}$$

$$d_4 = -p\left[2\Delta M (q_{P_2} q_{S_1} + p^2) - \Delta\rho\right], \tag{44}$$

$$c_1 = -2p^2 \Delta M (q_{P_1} + q_{P_2}) - (\rho_1 q_{P_2} - \rho_2 q_{P_1}), \tag{45}$$

$$c_2 = \left[2p^2 \Delta M (q_{S_1} + q_{S_2}) + (\rho_1 q_{S_2} - \rho_2 q_{S_1})\right], \tag{46}$$

$$c_3 = p\left[2\Delta M (q_{P_1} q_{S_2} - p^2) + \Delta\rho\right], \tag{47}$$

$$c_4 = p\left[2\Delta M (q_{P_2} q_{S_1} - p^2) + \Delta\rho\right]. \tag{48}$$

We can write the differences in complex moduli as a sum of the differences in elastic shear modulus plus an imaginary part

$$\Delta M = \Delta\mu + i\Delta\mu_A, \tag{49}$$

where the real part is given by

$$\Delta\mu_E = \mu_{E2} - \mu_{E1} = \rho_2 V_{SE_2}^2 - \rho_1 V_{SE_1}^2, \tag{50}$$

and imaginary part by

$$\Delta\mu_A = Q_{S_2}^{-1}\mu_{E2} - Q_{S_1}^{-1}\mu_{E1} = \rho_2 V_{SE_2}^2 Q_{S_2}^{-1} - \rho_1 V_{SE_1}^2 Q_{S_1}^{-1}. \tag{51}$$

By having the exact reflection coefficients, and the linearization tools for viscoelastic properties, we are ready to calculate the approximate reflectivities.

LINEARIZATION OF REFLECTIVITY

In this section we derive the linearized form of the P-to-P, P-to-SI and SI-to-SI reflection coefficients. For a low-loss viscoelastic medium we follow the Aki and Richards Aki and Richards (2002) approach which is based on the assumption of low contrast in both elastic and anelastic properties. For a viscoelastic medium, linearized coefficients are functions of the averages of elastic and anelastic properties across the interface and fractional changes in properties. The main goal of reflection seismology is to estimate the density, P- and S-wave velocities and corresponding quality factors of the earth layers from recorded seismic data. Amplitude variation with offset (AVO) analysis is based on the analytic expressions for reflection coefficients in elastic media. If anelasticity is present it is modified to a complex quantity whose real part is the elastic reflection coefficients. The exact form of the reflectivities are too complicated to extract intuitively much information about the physical properties of the subsurface earth. Practically for most reflecting interfaces in seismology the change in the elastic and anelastic properties are small, so that we can linearize the reflectivities in terms of perturbations of earth properties, defined as the ratio of the difference to the average of the properties of the contiguous layers.

P-to-P reflection coefficient

In this section we derive the first order approximation to the P-to-P reflection coefficient. Let us consider the case that an inhomogeneous P-wave hits the boundary of a slightly different low-loss viscoelastic medium. In this case the reflected P-wave is also an inhomogeneous wave. All complex quantities and expressions we have defined thus far include a first order contribution from the attenuation factor Q^{-1}. As a result, in the low-loss approximation any term in (38) which includes two imaginary parts in a product is negligible.

From Eqs. (47), (43), (48) and (44) we notice that $c_3 d_4$ and $d_3 d_4$ are in second order in the perturbations, which can be ignored in the first order approximation. If we keep first order terms only, the P-to-P reflection (38) reduces to

$$({}^{\downarrow}\mathrm{PP}^{\uparrow}) = \frac{c_1}{d_1} = \frac{2p^2 \Delta M (q_{P_1} + q_{P_2}) + (\rho_1 q_{P_2} - \rho_2 q_{P_1})}{2p^2 \Delta M (q_{P_1} - q_{P_2}) - (\rho_1 q_{P_2} + \rho_2 q_{P_1})}. \tag{52}$$

From the above equation we can not determine intuitively the influence of the change in a particular elastic or anelastic parameter on the reflectivity. We next linearize the above reflection coefficient according to the low contrast approximation. First if we expand (52) in terms of ΔM to first order we have

$$({}^{\downarrow}\mathrm{PP}^{\uparrow}) = ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{FF}} + ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{M}}, \tag{53}$$

where the first term is related to the reflectivity of the fluid-fluid interface

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{FF}} = \frac{1}{2}\left(\frac{\Delta\rho}{\rho} - \frac{\Delta q_{\mathrm{P}}}{q_{\mathrm{P}}}\right), \tag{54}$$

and the second term is related to the change in complex modulus M

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{M}} = -2p^2 \frac{\Delta M}{\rho}. \tag{55}$$

The linearization of the fluid-fluid reflectivity results

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{FF}} = ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{FF}}_{\mathrm{E}} + i({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{FF}}_{\mathrm{A}}, \tag{56}$$

where the first term, which is the real part of the $({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{FF}}$, is the reflectivity for a fluid-fluid interface for a non attenuative medium

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{FF}}_{\mathrm{E}} = \frac{1}{2}\frac{\Delta\rho}{\rho} + \frac{1}{2\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{PE}}}{V_{\mathrm{PE}}}, \tag{57}$$

and the anelastic part which induced by the change in P-wave velocity and P-wave quality factor is given by

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{FF}}_{\mathrm{A}} = \frac{1}{2\cos^2\theta} Q_{\mathrm{P}}^{-1}\left(\tan\theta_{\mathrm{P}}\tan\delta_{\mathrm{P}}\frac{\Delta V_{\mathrm{PE}}}{V_{\mathrm{PE}}} - \frac{1}{2}\frac{\Delta Q_{\mathrm{P}}}{Q_{\mathrm{P}}}\right). \tag{58}$$

Comparing the fluid-fluid reflectivity for elastic and viscoelastic media it can be seen that, in the elastic case, when the contrast in P-wave velocity is zero, reflectivity depends only upon the change in the density and is independent of the incidence angle even if the contrast in P-wave velocity is zero. On the other hand for a viscoelastic medium, the reflectivity is angle dependent. Consider the term in equation (55), which is the contribution of the change of M to the reflectivity. The real and imaginary parts of the perturbation in complex moduli are given by

$$\Delta\mu_E = \rho V_{\mathrm{SE}}^2 \left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}\right], \tag{59}$$

$$\Delta\mu_A = \rho Q_{\mathrm{S}}^{-1} V_{\mathrm{SE}}^2 \left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}} - \frac{\Delta Q_{\mathrm{S}}}{Q_{\mathrm{S}}}\right). \tag{60}$$

Inserting the above relations in (55), we arrive at

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{M}} = ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{M}}_{\mathrm{E}} + i({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{M}}_{\mathrm{A}}. \tag{61}$$

Where the real part, which is related to the case where the attenuation is zero in the medium, is given by

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{M}}_{\mathrm{E}} = -2\sin^2\theta_P \left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2 \left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}\right], \tag{62}$$

and the imaginary part by

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{M}}_{\mathrm{A}} = -2\sin^2\theta_P \left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2 \left\{\left[\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}\right]\left(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1}\left[1 - \frac{\tan\delta_{\mathrm{P}}}{\tan\theta_{\mathrm{P}}}\right]\right) - Q_{\mathrm{S}}^{-1}\frac{\Delta Q_{\mathrm{S}}}{Q_{\mathrm{S}}}\right\}. \tag{63}$$

Evidently contrasts in complex moduli M affect the reflectivity only for nonzero offsets. Finally, the P-to-P reflectivity can be rewritten into

$$({}^{\downarrow}\mathrm{PP}^{\uparrow}) = ({}^{\downarrow}\mathrm{PP}^{\uparrow})_{\mathrm{E}} + i({}^{\downarrow}\mathrm{PP}^{\uparrow})_{\mathrm{A}}, \tag{64}$$

where the imaginary part is given by

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})_{\mathrm{A}} = ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\rho}_{\mathrm{A}} + ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{V_{PE}}}_{\mathrm{A}} + ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{V_{SE}}}_{\mathrm{A}} + ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{Q_P}}_{\mathrm{A}} + ({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{Q_S}}_{\mathrm{A}}, \tag{65}$$

with the density component

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\rho}_{\mathrm{A}} = -\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2 \left(2(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1})\sin^2\theta_P + Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}}\sin 2\theta_{\mathrm{P}}\right)\frac{\Delta\rho}{\rho}, \tag{66}$$

the P-wave velocity component

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{V_{PE}}}_{\mathrm{A}} = Q_{\mathrm{P}}^{-1}\frac{\tan\theta_{\mathrm{P}}\tan\delta_{\mathrm{P}}}{2\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{PE}}}{V_{\mathrm{PE}}}, \tag{67}$$

the S-wave velocity component

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{V_{SE}}}_{\mathrm{A}} = -2\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2 \left(2(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1})\sin^2\theta_P + Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}}\sin 2\theta_{\mathrm{P}}\right)\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}, \tag{68}$$

the P-wave quality factor component

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})^{\mathrm{Q_P}}_{\mathrm{A}} = -\frac{1}{4\cos^2\theta}Q_{\mathrm{P}}^{-1}\frac{\Delta Q_{\mathrm{P}}}{Q_{\mathrm{P}}}, \tag{69}$$

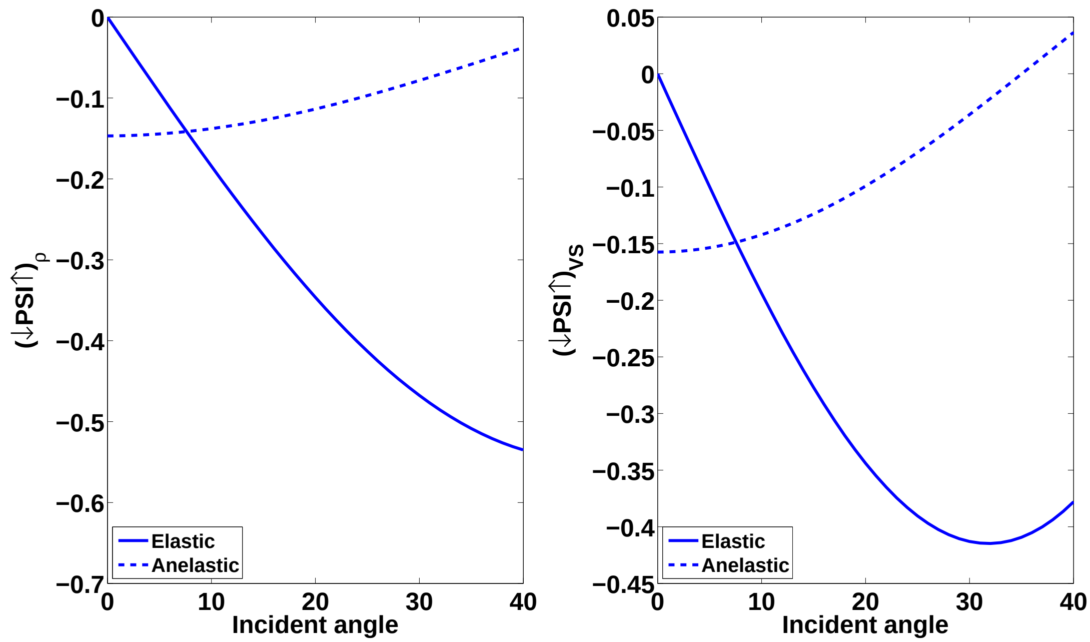

FIG. 3. Elastic and anelastic density(left) and S-wave velocity(right) components of the P-to-P reflection coefficient, for incident inhomogeneous P-wave to inhomogeneous reflected P-wave versus of incident angle θ_{P}, for average values $\delta_{\mathrm{P}} = \delta_{\mathrm{S}} = 60°$. Average of P-wave quality factor for two layer is to be 13 and for S-wave quality factor is 11. Also the average S-to P-wave velocity ratio is chosen to be $.58$. Dash line is for anelastic part and solid line for elastic part.

and the S-wave quality factor component

$$(^{\downarrow}\mathrm{PP}^{\uparrow})_{\mathrm{A}}^{\mathrm{Q_S}} = 2\sin^2\theta_P \left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2 Q_{\mathrm{S}}^{-1}\frac{\Delta Q_{\mathrm{S}}}{Q_{\mathrm{S}}}. \tag{70}$$

The real part is

$$(^{\downarrow}\mathrm{PP}^{\uparrow})_{\mathrm{E}} = \frac{1}{2}\left(\frac{\Delta\rho}{\rho} + \frac{1}{\cos^2\theta_{\mathrm{P}}}\frac{\Delta V_{\mathrm{PE}}}{V_{\mathrm{PE}}}\right) - 2\sin^2\theta_P\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2\left(\frac{\Delta\rho}{\rho} + 2\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}\right). \tag{71}$$

$(^{\downarrow}\mathrm{PP}^{\uparrow})_{\mathrm{E}}$ is the P-to-P reflection coefficient for a low contrast interface between two isotropic elastic layered media. The real part of the linearized P-to-P reflectivity for two slightly different low-loss viscoelastic media is the P-to-P reflection coefficient induced by a contrast in elastic properties. The imaginary part is caused by both elastic and/or anelastic contrasts between the two layeres. As a consequence, even if the quality factors for P- and S-waves do not change between the two layers, the contrasts in elastic properties can still contribute an imaginary part. At normal incidence,

$$(^{\downarrow}\mathrm{PP}^{\uparrow}) = \frac{\Delta\rho}{2\rho} + \frac{\Delta V_{\mathrm{PE}}}{2V_{\mathrm{PE}}} - \frac{i}{4}Q_{\mathrm{P}}^{-1}\frac{\Delta Q_{\mathrm{P}}}{Q_{\mathrm{P}}}. \tag{72}$$

So, at normal incidence the reflectivity is affected by the by contrast in the P-wave quality factor. The relative change in density and P-wave velocity have similar influence on the normal incidence reflection coefficient. Fig.3 shows the elastic and anelastic parts of Amplitude Variation with Angle (AVA) for density and S-wave velocity parts. Generally, the S-wave velocity contrasts contribute more to AVA variations.

LINEARIZED P-TO-SI REFLECTION

In this section we derive the converted P-to-SI reflection coefficient for a two-layered low-loss viscoelastic medium with small contrast in material properties across the boundary. For an isotropic viscoelastic medium, the converted P-to-SI wave amplitude variation patterns reveal the changes in density, S-wave velocity and quality factor. When the incidence wave is an inhomogeneous P-wave, the reflected wave can be either an inhomogeneous P- or SI-wave. In contrast to the elastic case, the linearization is more complicated and the linearized result includes the terms related to the change in S-wave quality factors. Under the low-contrast medium assumption, the P-to-SI reflection coefficient in equation (39)reduces to

$$(^{\downarrow}\mathrm{PSI}^{\uparrow}) = -\left(\frac{V_{\mathrm{P1}}}{V_{\mathrm{S1}}}\right)\frac{c_1d_3 + c_3d_1}{d_1d_2} = -\left(\frac{V_{\mathrm{P1}}}{V_{\mathrm{S1}}}\right)\frac{p}{q_S}\frac{[2(q_{\mathrm{P}}q_S - p^2)\Delta M + \Delta\rho]}{2\rho}. \tag{73}$$

In the low-loss approximation we have

$$\frac{p}{q_{\mathrm{S}}} = \tan\theta_S\left(1 + iQ_{\mathrm{S}}^{-1}\frac{\tan\delta_S}{\sin 2\theta_S}\right), \tag{74}$$

where q_S, is the average of the vertical slowness for S-wave. To produce a form of the reflectivity that explicitly shows its dependency upon medium property perturbations, we first consider the multiplication of the P- and S-wave vertical slowness vectors

$$q_{\mathrm{P}}q_{\mathrm{S}} = \frac{1}{V_{\mathrm{PE}}V_{\mathrm{SE}}}\left[\cos\theta_{\mathrm{P}}\cos\theta_S\left(1 - \frac{i}{2}(Q_{\mathrm{P}}^{-1} + Q_{\mathrm{S}}^{-1})\right) - \right.$$

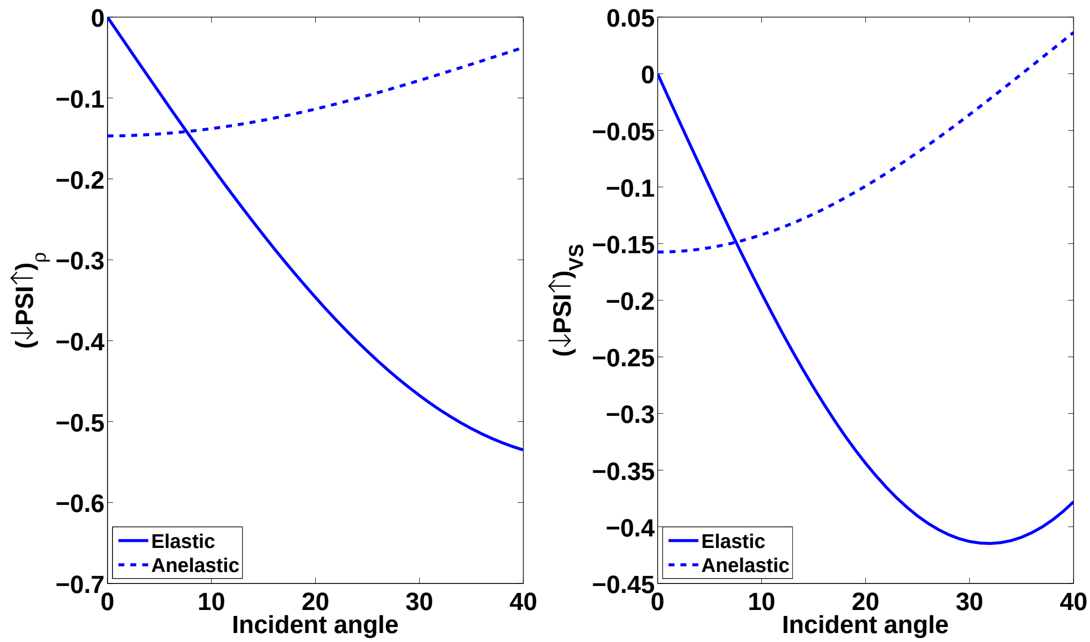

FIG. 4. Elastic and anelastic density(left) and S-wave velocity(right) components of the P-to-SI reflection coefficient, for incident inhomogeneous P-wave to inhomogeneous reflected SI-wave versus of incident angle θ_{P}, for average values $\delta_{\mathrm{P}} = \delta_{\mathrm{S}} = 60°$. Average of P-wave quality factor for two layer is to be 13 and for S-wave quality factor is 11. Also the average S-to P-wave velocity ratio is chosen to be $.58$. Dash line is for anelastic part and solid line for elastic part.

$$\frac{i}{2}(Q_{\mathrm{S}}^{-1}\tan\delta_S\cos\theta_{\mathrm{P}}\sin\theta_S+Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}}\cos\theta_S\sin\theta_{\mathrm{P}})\Big], \tag{75}$$

and square of the ray parameter

$$p^2=\frac{1}{V_{\mathrm{PE}}V_{\mathrm{SE}}}\left[\sin\theta_{\mathrm{P}}\sin\theta_S\left(1-\frac{i}{2}(Q_{\mathrm{P}}^{-1}+Q_{\mathrm{S}}^{-1})\right)+\right.$$

$$\left.\frac{i}{2}(Q_{\mathrm{S}}^{-1}\tan\delta_S\cos\theta_S\sin\theta_{\mathrm{P}}+Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}}\cos\theta_{\mathrm{P}}\sin\theta_S)\right]. \tag{76}$$

Using the perturbation in complex moduli in equations (50) and (51), we arrive at the linearized P-to-SI reflectivity function

$$({}^{\downarrow}\mathrm{PSI}^{\uparrow})=({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{E}}+i({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{A}}, \tag{77}$$

where the imaginary part is related to the change in S-wave quality factor, density and S-wave velocity

$$({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{A}}=({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{A}}^{\rho}+({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{A}}^{\mathrm{V_{SE}}}+({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{A}}^{\mathrm{Q_S}}, \tag{78}$$

with the density component

$$\begin{aligned}({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{A}}^{\rho}=&-\frac{1}{2}\tan\theta_S\left[\frac{1}{2}(Q_P^{-1}-Q_S^{-1})+Q_S^{-1}\frac{\tan\delta_S}{\sin 2\theta_S}\right]\frac{V_{PE}}{V_{SE}}\frac{\Delta\rho}{\rho}\\&-\tan\theta_S\left[\frac{\tan\delta_S}{\sin 2\theta_S}\cos(\theta_P+\theta_S)\right]Q_S^{-1}\frac{\Delta\rho}{\rho}\\&+\frac{1}{2}\tan\theta_S\left[\sin(\theta_P+\theta_S)(Q_S^{-1}\tan\delta_S+Q_P^{-1}\tan\delta_P)\right]\frac{\Delta\rho}{\rho}\end{aligned} \tag{79}$$

the S-wave velocity component

$$\begin{aligned}({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{A}}^{\mathrm{V_{SE}}}=&-2\tan\theta_S\left[\frac{\tan\delta_S}{\sin 2\theta_S}\cos(\theta_P+\theta_S)\right]Q_S^{-1}\frac{\Delta V_{SE}}{V_{SE}}\\&+\tan\theta_S\left[\sin(\theta_P+\theta_S)(Q_S^{-1}\tan\delta_S+Q_P^{-1}\tan\delta_P)\right]\frac{\Delta V_{SE}}{V_{SE}}\end{aligned} \tag{80}$$

and the S-wave quality factor component

$$({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{A}}^{\mathrm{Q_S}}=Q_S^{-1}\tan\theta_S\cos(\theta_P+\theta_S)\frac{\Delta Q_S}{Q_S}. \tag{81}$$

In addition, $({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{E}}$ is the reflectivity for non-attenuative medium or the P-to-SV reflectivity. It is given by

$$({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{E}}=-\tan\theta_S\left(\cos(\theta_{\mathrm{P}}+\theta_S)+\frac{1}{2}\frac{V_{\mathrm{PE}}}{V_{\mathrm{SE}}}\right)\frac{\Delta\rho}{\rho}-2\tan\theta_S\cos(\theta_{\mathrm{P}}+\theta_S)\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}. \tag{82}$$

In the above expressions, $\theta_{\mathrm{P}}(\theta_{\mathrm{S}})$ is the average of angles of incidence and transmission for P(SI)-wave; θ_{S} can be calculated from θ_{P} using Snell's law; ρ, V_{PE}, V_{SE}, $\mathrm{Q_S}$, $\mathrm{Q_P}$, δ_{P} and δ_{S} are the average quantities. Unlike the P-to-P reflection coefficient, the P-to-SI reflection coefficient does not depend on the contrasts in the P-wave velocity and its quality factor.

Similarly to the P-to-P reflection case, $({}^{\downarrow}\mathrm{PSI}^{\uparrow})_{\mathrm{E}}$ denotes the low contrast reflection coefficient at an interface separating two elastic media. Equation (77) represents the P-to-SI reflection coefficient for a low contrast interface separating two arbitrary low-loss viscoelastic media. By inspecting the above equations, at normal incidence, the reflection coefficient is evidently not affected by either elasticity or anelasticity. Approximate SI-to-P reflection coefficient is similar to that of the P-to-SI coefficient; the only difference is that the exchange of rule between P and SI waves. Fig. 4 displays the elastic and anelastic parts of amplitude versus angle (AVA) for density and S-wave velocity parts.

SI-TO-SI REFLECTION

It has been shown that an SI-wave can be reflected or scattered to either P- or SI-waves, but can not be converted to an SII-wave with a linear polarization. SI-to-P reflectivity is very similar to the P-to-SI case, so we will not consider this case here. Let us consider the reflection of the SI-to-SI wave. In the first order approximation the exact SI-to-SI reflectivity in equation (40) reduces to

$$({}^{\downarrow}\mathrm{SISI}^{\uparrow}) = \frac{c_2}{d_2} = \frac{2p^2\Delta M(q_{S_1}+q_{S_2})+(\rho_1 q_{S_2}-\rho_2 q_{S_1})}{2p^2\Delta M(q_{S_1}-q_{S_2})-(\rho_1 q_{S_2}+\rho_2 q_{S_1})}. \tag{83}$$

After applying the linearization procedure we arrive at

$$({}^{\downarrow}\mathrm{SISI}^{\uparrow}) = ({}^{\downarrow}\mathrm{SISI}^{\uparrow})_E + i({}^{\downarrow}\mathrm{SISI}^{\uparrow})_A, \tag{84}$$

where the elastic part is given by

$$({}^{\downarrow}\mathrm{SISI}^{\uparrow})_E = \frac{1}{2}\left(1-4\sin^2\theta_S\right)\frac{\Delta\rho}{\rho}+\frac{1}{2\cos^2\theta_{\mathrm{S}}}(1-2\sin^2 2\theta_S)\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}, \tag{85}$$

and the anelastic term is

$$({}^{\downarrow}\mathrm{SISI}^{\uparrow})_A = ({}^{\downarrow}\mathrm{SISI}^{\uparrow})_A^{\rho} + ({}^{\downarrow}\mathrm{SISI}^{\uparrow})_A^{V_{\mathrm{SE}}} + ({}^{\downarrow}\mathrm{SISI}^{\uparrow})_A^{Q_S}, \tag{86}$$

with the density component being

$$({}^{\downarrow}\mathrm{SISI}^{\uparrow})_A^{\rho} = -Q_{\mathrm{S}}^{-1}\sin 2\theta_S\tan\delta_{\mathrm{S}}\frac{\Delta\rho}{\rho}, \tag{87}$$

the S-wave velocity component

$$({}^{\downarrow}\mathrm{SISI}^{\uparrow})_A^{V_{\mathrm{SE}}} = -\frac{\tan\theta_S}{2\cos^2\theta_S}\left(1-8\cos^4\theta_S\right)\tan\delta_S Q_{\mathrm{S}}^{-1}\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}, \tag{88}$$

and the S-wave quality factor component being

$$({}^{\downarrow}\mathrm{SISI}^{\uparrow})_A^{Q_S} = \frac{1}{4\cos^2\theta_S}Q_{\mathrm{S}}^{-1}\left(2\sin^2 2\theta_S-1\right)\frac{\Delta Q_{\mathrm{S}}}{Q_{\mathrm{S}}}. \tag{89}$$

In Fig.5, we plot the density and S-wave velocity components of the reflectivity versus the incident angle. The solid line represents for the elastic part and dashed line is the anelastic part.

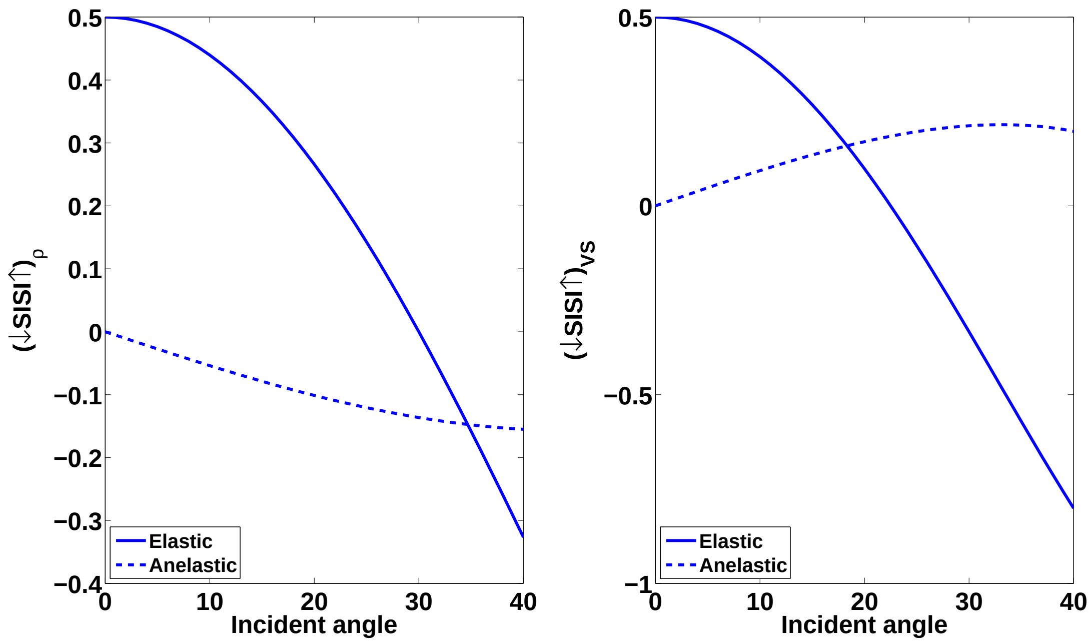

FIG. 5. Elastic and anelastic density(left) and S-wave velocity(right) components of the reflection coefficient SI-to-SI, for incident inhomogeneous SI-wave to inhomogeneous reflected SI-wave versus of incident angle $\theta_{\rm P}$, for average value $\delta_{\rm S} = 60°$. Average of P-wave quality factor for two layer is to be 13 and for S-wave is 11. Also the average S-to P-wave velocity ratio is chosen to be $.58$. Dash line is for anelastic part and solid line for elastic part.

THE RELATIONSHIP BETWEEN THE REFLECTIVITY REFLECTIVITY AND THE VISCOELASTIC SCATTERING POTENTIAL

Here we relate the linearized forms for viscoelastic reflection and conversion to recent results concerning the general problem of scattering of viscoelastic waves. The reflectivity picture and the volume scattering picture are not precisely equivalent (see Figure 6); however under the assumption of small angle and small contrasts across the reflecting boundary the two are consistent, and indeed are mathematically equivalent.

The scattering potential (see, e.g., Stolt and Weglein (2012); Weglein et al. (2003); Beylkin and Burridge (1990)) enters into a modeling of wave interaction with heterogenous media either through the Born approximation which neglects nonlinearity in the wave/medium relationship, or through the full scattering series, in which amplitudes and phases of waves accommodate large and extended perturbation (e.g.,Weglein et al. (2003); Innanen (2009)), and events whose propagation histories have introduced more that one subsurface reflection, like multiples are incorporated (Weglein and Dragoset, 2005). One instance of the scattering potential arises in the Born approximation, thus analysis of the potential in isolation qualitatively "feels like" analysis of the Born approximation. It can be this, but we emphasize the potential is description of the full, nonlinear problem also.

Generally the relationship between the scattering potential and reflectivity function is given by (Beylkin and Burridge, 1990)

$$({}^{\downarrow}\mathrm{IR}^{\uparrow}) = -\frac{1}{2V_{\mathrm{R}}^2 q_{\mathrm{R}}(q_{\mathrm{R}}+q_{\mathrm{I}})}{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}, \tag{90}$$

where index I refers to the type of the incidence wave and R indicates the type of reflected wave and V_{R} is the wave velocity corresponding to the wave type R. A similar relation applies to the vertical slownesses, q_{R} and q_{I}. Additionally ${}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}$, is the scattering potential for the incidence wave type I and scattered type R. In the elastic case we have

$$({}^{\downarrow}\mathrm{IR}^{\uparrow})_{\mathrm{E}} = -\frac{\sin\theta_{\mathrm{I}}}{2\cos\theta_{\mathrm{R}}\sin(\theta_{\mathrm{I}}+\theta_{\mathrm{R}})}{}^{\mathrm{I}}_{\mathrm{R}}\mathbb{V}_{\mathrm{E}}. \tag{91}$$

Where θ_{I} is the phase angle for wave type I and θ_{R} is the phase angle for wave type R. For example if the reflected wave is a P-wave, $\theta_{\mathrm{R}} \equiv \theta_{\mathrm{P}}$, which can be interpreted either as a phase angle for the incidence P-wave or as a average of incidence and transmitted P-waves. In the next section we obtain one-to-one relationships between the scattering potential and reflectivity functions.

P-to-P scattering potential

In this case the incidence and reflected waves are P-type; as a result, $\theta_{\mathrm{I}} = \theta_{\mathrm{R}} = \theta_{\mathrm{P}}$. θ_{P} interpreted as the average of phase angles of the incident and transmitted P-waves. Since for the low contrast there is only a small difference between the incident and transmitted angles, we can say that θ_{P} is the phase angle of the incidence wave. However in the development of linearization for the reflectivity we assumed that their difference is not zero. For elastic

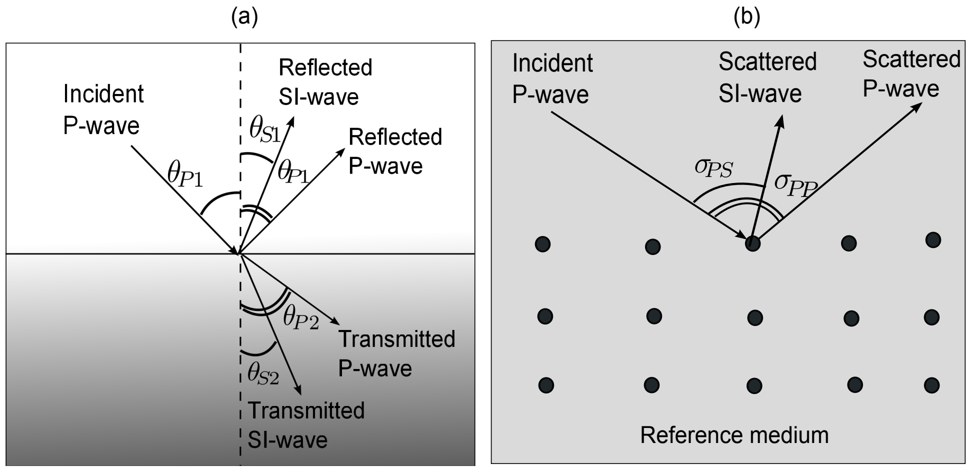

FIG. 6. Comparing of the reflecting half-space vs Born approximation frameworks. (a) The boundary is assumed to involved welded contact between two media whose properties differ only slightly. Incident, reflected and transsmited rays are related by Snell's law;interface normal helps define ray angles. (b) Reference medium is perturbed by one or more volume scattering inclusions; ray angles are defined in terms of the opening angle between incident and scattered rays. $\sigma_{\mathrm{PP}} = 2\theta_{\mathrm{P}}$, is the opening angle between the incident and scattered P-waves, where θ_{P} is the average of incident and transmitted angles. $\sigma_{\mathrm{PS}} = \theta_{\mathrm{P}} + \theta_{\mathrm{S}}$, is the opening angle between the incident P-wave and scattered SI-wave, where θ_{S} is the average of incident and transmitted SI-waves.

case the relationship between reflectivity and scattering potential reduces to

$$({}^{\downarrow}\mathrm{PP}^{\uparrow})_{\mathrm{E}} = -\frac{1}{4\cos^2\theta_{\mathrm{P}}}\,{}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}_{\mathrm{E}}. \tag{92}$$

Let us now consider to the viscoelastic case, specifically one in which an inhomogeneous P-wave reflects to an inhomogeneous P-wave. In this case the relation between the reflectivity and scattering potential is given by

$${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V} = -4V_{\mathrm{P}}^2 q_{\mathrm{P}}^2({}^{\downarrow}\mathrm{PP}^{\uparrow}), \tag{93}$$

or

$${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V} = -4\cos^2\theta_{\mathrm{P}}(1 - iQ_{\mathrm{P}}^{-1}\tan\theta_{\mathrm{P}}\tan\delta_{\mathrm{P}})({}^{\downarrow}\mathrm{PP}^{\uparrow}). \tag{94}$$

After doing some algebra we find that the reflectivity we derived in and around equation (88), upon substitution into equation (101), results in a scattering potential

$${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V} = {}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}_{\mathrm{E}} + i\,{}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}_{\mathrm{A}}, \tag{95}$$

where the elastic scattering potential for PP-mode is

$${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}_{\mathrm{E}} = \left[-1 - \cos\sigma_{\mathrm{PP}} + 2\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2\sin^2\sigma_{\mathrm{PP}}\right]\frac{\Delta\rho}{\rho} - 2\frac{\Delta V_{\mathrm{PE}}}{V_{\mathrm{PE}}} + 4\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2\sin^2\sigma_{\mathrm{PP}}\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}, \tag{96}$$

and the anelastic part

$${}^{\mathrm{P}}_{\mathrm{P}}\mathbb{V}_{\mathrm{A}} = \left[Q_{\mathrm{P}}^{-1}\sin\sigma_{\mathrm{PP}}\tan\delta_{\mathrm{P}} + 2\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)^2\left\{\sin^2\sigma_{\mathrm{PP}}(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1}) + Q_{\mathrm{P}}^{-1}\sin 2\sigma_{\mathrm{PP}}\tan\delta_{\mathrm{P}}\right\}\right]\frac{\Delta\rho}{\rho}, \tag{97}$$

$$+4\left(\frac{V_{\rm SE}}{V_{\rm PE}}\right)^2\left[\sin^2\sigma_{\rm PP}(Q_{\rm S}^{-1}-Q_{\rm P}^{-1})+Q_{\rm P}^{-1}\sin 2\sigma_{\rm PP}\tan\delta_{\rm P}\right]\frac{\Delta V_{\rm SE}}{V_{\rm SE}} \tag{98}$$

$$-2Q_{\rm S}^{-1}\left(\frac{V_{\rm SE}}{V_{\rm PE}}\right)^2\sin^2\sigma_{\rm PP}\frac{\Delta Q_{\rm S}}{Q_{\rm S}} \tag{99}$$

$$+Q_{\rm P}^{-1}\frac{\Delta Q_{\rm P}}{Q_{\rm P}}, \tag{100}$$

Where, $\sigma_{\rm PP} = 2\theta_{\rm P}$, is the opening angle between the incidence and reflected waves. The above expression is the same as that obtained using the volume scattering formalism (Moradi and Innanen, 2015). Thus our current linearization of reflectivity is consistent with the more general scattering picture.

SI-to-SI scattering potential

Similar to the P-to-P case, the relation between the scattering potential for SI-to-SI wave and its corresponding linearized reflection is given by

$$^{\rm SI}_{\rm SI}\mathbb{V} = -4\cos^2\theta_S(1-iQ_{\rm S}^{-1}\tan\theta_{\rm S}\tan\delta_{\rm S})(^{\downarrow}{\rm SISI}^{\uparrow}). \tag{101}$$

The scattering potential for the scattering of the SI-wave to SI-wave is determined to be

$$^{\rm SI}_{\rm SI}\mathbb{V} = {}^{\rm SV}_{\rm SV}\mathbb{V}_{\rm E} + i^{\rm SI}_{\rm SI}\mathbb{V}_{\rm A}. \tag{102}$$

The real part is the elastic scattering potential for scattering of SV-wave to SV-wave

$$^{\rm SV}_{\rm SV}\mathbb{V}_{\rm E} = -\left(\cos(2\sigma_{\rm SS})+\cos\sigma_{\rm SS}\right)\frac{\Delta\rho}{\rho} - 2\cos(2\sigma_{\rm SS})\frac{\Delta V_{\rm SE}}{V_{\rm SE}}, \tag{103}$$

and the anelastic part is given by

$$^{\rm SI}_{\rm SI}\mathbb{V}_{\rm A} = Q_{\rm S}^{-1}\left(\sin\sigma_{\rm SS}+2\sin(2\sigma_{\rm SS})\right)\tan\delta_S\frac{\Delta\rho}{\rho} \tag{104}$$

$$+4Q_{\rm S}^{-1}\sin(2\sigma_{\rm SS})\tan\delta_S\frac{\Delta V_{\rm SE}}{V_{\rm SE}} - \cos(2\sigma_{\rm SS})Q_{\rm S}^{-1}\frac{\Delta Q_{\rm S}}{Q_{\rm S}}, \tag{105}$$

Where, $\sigma_{\rm SS} = 2\theta_{\rm S}$, is the opening angle between the incidence and scattered waves, which is the scattering potential obtained using the Born approximation.

P-to-SI scattering potential

First the relation between the reflectivity and scattering potential is given by

$$^{\rm P}_{\rm SI}\mathbb{V} = -2V_{\rm S}^2 q_{\rm S}(q_{\rm S}+q_{\rm P})(^{\downarrow}{\rm PSI}^{\uparrow}). \tag{106}$$

The scattering potential for P-to-SI is, consequently,

$$^{\rm P}_{\rm SI}\mathbb{V} = {}^{\rm P}_{\rm SI}\mathbb{V}_{\rm E} + i^{\rm P}_{\rm SI}\mathbb{V}_{\rm A}. \tag{107}$$

where the elastic part of the scattering potential ${}^{\mathrm{P}}_{\mathrm{SI}}\mathbb{V}_{\mathrm{E}}$ is given by

$$
{}^{\mathrm{P}}_{\mathrm{SI}}\mathbb{V}_{\mathrm{E}} = -\left[\sin\sigma_{\mathrm{PS}} + \left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)\sin 2\sigma_{\mathrm{PS}}\right]\frac{\Delta\rho}{\rho} - \left[\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)\sin 2\sigma_{\mathrm{PS}}\right]\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}}. \tag{108}
$$

and the anelastic part is given by

$$
\begin{aligned}
{}^{\mathrm{P}}_{\mathrm{SI}}\mathbb{V}_{\mathrm{A}} = -\frac{1}{2}\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)\Big\{\sin 2\sigma_{\mathrm{PS}}(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1}) + \\
\left[2\cos 2\sigma_{\mathrm{PS}} + \left(\frac{V_{\mathrm{PE}}}{V_{\mathrm{SE}}}\right)\cos\sigma_{\mathrm{PS}}\right](Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{S}} + Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}})\Big\}\frac{\Delta\rho}{\rho} \\
-\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)\Big\{\sin 2\sigma_{\mathrm{PS}}(Q_{\mathrm{S}}^{-1} - Q_{\mathrm{P}}^{-1}) + 2\cos 2\sigma_{\mathrm{PS}}(Q_{\mathrm{S}}^{-1}\tan\delta_{\mathrm{S}} + Q_{\mathrm{P}}^{-1}\tan\delta_{\mathrm{P}})\Big\}\frac{\Delta V_{\mathrm{SE}}}{V_{\mathrm{SE}}} \\
+\left(\frac{V_{\mathrm{SE}}}{V_{\mathrm{PE}}}\right)Q_{\mathrm{S}}^{-1}\sin 2\sigma_{\mathrm{PS}}\frac{\Delta Q_{\mathrm{S}}}{Q_{\mathrm{S}}}.
\end{aligned} \tag{109}
$$

Here, the opening angle between the incidence P-wave and reflected SI-wave is $\sigma_{\mathrm{PS}} = \theta_{\mathrm{P}} + \theta_{\mathrm{S}}$. Also, θ_{P} for a welded boundary is the average of the incidence P-wave and transmitted P-wave; the same interpretation applies for θ_{S}.

CONCLUSION

Amplitude variation with offset (AVO) or amplitude variation with angle (AVA) analysis is a study of the effects of change in medium properties and incident angle on the reflection coefficients as the contrast between two layer is weak. Even in the case of an isotropic elastic medium the exact equations for the reflection coefficients are sufficiently complicated that the effects of changes in medium properties and dependency on the incidence angle is not explicitly clear. When attenuation is added to medium as in a viscoelastic case, the problem gets still more complicated. In this case, besides the elastic properties and phase angle, reflectivity is sensitive to change in the anelastic quantities and attenuation angles. Since in practical cases, attenuation is often weak, the reflection coefficient takes on a more form. In our linearization besides the assumption of weak contrasts in elastic and anelastic properties, we applied the additional assumption of weak attenuation, i.e., what is termed a low-loss medium, in both half spaces.

Linearized forms of PP, PSI and SISI reflection coefficients for low contrast interfaces separating two arbitrary low-loss viscoelastic media were derived. The linearized viscoelastic reflection coefficient we derived relate the AVO response to the anelastic parameters. It is shown that the reflectivity not only depends upon the perturbations in elastic properties, but also on perturbations in quality factors for P- and S-waves. Also using the viscoelastic Snell's law we show that the transmitted and reflected P- and S-waves attenuation angles can be expressed in terms of incidence angle and incidence attenuation angles. To derive the reflectivities, we linearized Snell's law for a two layer viscoelastic media and show that in the linearized reflectivity only the average of attenuation angle effects the reflectivity. Also we showed that the linearized reflectivities can be transformed to the scattering potential obtained using the Born approximation.

To model the anelasticity in a medium linear viscoelasticity is used. Plane waves are generally inhomogeneous, where the attenuation and propagation are not in the same direction. The elastic reflectivity can be obtained in the limit that attenuation quantities Q go to zero. If all parameters related to the anelasticity go to zero the viscoelastic reflections reduced to the linearized elastic isotropic reflection coefficients obtained by Aki and Richards Aki and Richards (2002).

ACKNOWLEDGMENTS

The authors thank the sponsors of CREWES for continued support. This work was funded by CREWES industrial sponsors and NSERC (Natural Science and Engineering Research Council of Canada) through the grant CRDPJ 461179-13.

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