Comments on the paper of Clayton and Engquist (1977) – absorbing boundary conditions

P.F. Daley and E.S. Krebes

ABSTRACT

The paper by Clayton and Engquist (1977) is often quoted in the literature, almost exclusively when problems involving finite difference methods are being discussed when dealing with acoustic wave propagation and coupled P-S_V wave propagation in an elastic medium using finite differences or related methods. With the recent interest in perfectly matched layers (PML) methods it is often used as a bench mark with which to determine the numerical accuracy of this relatively new method (for example, Zhu and McMechan, 1991). This attention is for the most part based on one page of the 1977 paper. There are some cursory instructions on how to proceed to obtain paraxial approximations for the wave equations in the vicinity of finite, usually perfectly reflecting, boundaries and how to employ them. As the single page is followed by an appendix for its implementation, little thought has been paid to what has been said on that page. Here we would like to expand on that page for the information of others who wish to use this method for similar, usually more complex, problems and for its possible use in hybrid methods, where one or more of the spatial derivatives have been removed by integral transform methods. As others have questioned the authors on this topic, it was thought that this mild tutorial could be useful.

INTRODUCTION

It would probably be useful to begin this with a quote from Clayton and Engquist (1977):

"Paraxial approximations for the elastic wave equation analogous to those of the scalar wave equation can also be found. We cannot, however, perform the analysis by considering expansions of the dispersion relation because the differential equations for vector fields are not uniquely specified from their dispersion relations. Instead, we use the scalar case to provide a hint as to the general form of the paraxial approximation and fit the coefficients by matching to the full elastic wave equation." [Clayton and Engquist (1977)]

When considering the solution of hyperbolic systems of equations by finite difference methods, the problem of minimizing or eliminating reflections from the finite boundaries which are required to be introduced, even though the initial problem was infinite in its statement. Over several decades the paper by Clayton and Engquist (1977) has been the standard by which most other solutions to this are compared. As a consequence of requiring to employ the theory in the above paper, some preliminary thoughts that there may be a minor and at the request others using this theory, it was thought that a re-derivation of all contained in that paper was in order. The only error found was a minor typographic in the Appendix. However, as time was spent doing this, it was thought that some *gaps* in the theory of the paper be set forth.

THEORY

In a 2D isotropic homogeneous medium with the source term deleted, the expressions for the horizontal (u) and vertical (w) components of displacement have the form

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + \left(\alpha^2 - \beta^2\right)\frac{\partial w}{\partial x \partial z} + \beta^2 \frac{\partial^2 u}{\partial z^2} \tag{1}$$

$$\frac{\partial^2 w}{\partial t^2} = \beta^2 \frac{\partial^2 w}{\partial x^2} + \left(\alpha^2 - \beta^2\right)\frac{\partial^2 u}{\partial x \partial z} + \alpha^2 \frac{\partial^2 w}{\partial z^2} \tag{2}$$

where α and β are the compressional (P) and shear (S) velocities. The above may be written in matrix – vector form as

$$\mathbf{u}_{tt} = D_1 \mathbf{u}_{xx} + H \mathbf{u}_{xz} + D_2 \mathbf{u}_{zz} \tag{3}$$

with

$$\mathbf{u} = \left[u, w\right]^T \tag{4}$$

and the coefficient matrices as

$$D_1 = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix} \quad , \quad D_2 = \begin{bmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{bmatrix} \quad , \quad H = \left(\alpha^2 - \beta^2\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

Assume a plane wave solution of equation (3) of the form:

$$\mathbf{u} = \tilde{\mathbf{u}}\left[\exp\left(-i\omega t + ik_x x + ik_z z\right)\right] \tag{6}$$

In terms of pseudo differential operators the above results in

$$\begin{aligned} \mathbf{u}_t &\to \left(-i\omega\right)\tilde{\mathbf{u}} \\ \mathbf{u}_z &\to \left(ik_z\right)\tilde{\mathbf{u}} \\ \mathbf{u}_x &\to \left(ik_x\right)\tilde{\mathbf{u}} \end{aligned} \tag{7}$$

Substituting these into (3) results in

$$\left[I - D_1 \frac{\left(ik_x\right)^2}{\left(-i\omega\right)^2} - H \frac{\left(ik_x\right)\left(ik_z\right)}{\left(-i\omega\right)^2} - D_2 \frac{\left(ik_z\right)^2}{\left(-i\omega\right)^2}\right]\tilde{\mathbf{u}} = 0 \tag{8}$$

where $\tilde{\mathbf{u}}$ is the transform of $\mathbf{u}$, I is the 2×2 identity matrix and all other quantities were previously defined.

Zero order paraxial approximation:

Taking the first and fourth terms of equation (8), under the assumption that $\mathbf{u}\,\tilde{}\neq 0$, results in the following condition required to be valid

$$\left[\left(-i\omega\right)^2 - D_2\left(ik_z\right)^2\right] = 0\,. \tag{9}$$

Some basic manipulations of the above equation given here as

$$\begin{gathered}\left(-i\omega\right)^2 = D_2\left(ik_z\right)^2 \quad \rightarrow \quad \left(-i\omega\right) = \left(D_2\right)^{1/2}\left(ik_z\right) \\ \left(D_2\right)^{1/2} = \begin{bmatrix}\beta & 0 \\ 0 & \alpha\end{bmatrix} \quad \rightarrow \quad \left[\left(D_2\right)^{1/2}\right]^{-1} = \begin{bmatrix}1/\beta & 0 \\ 0 & 1/\alpha\end{bmatrix} = B_1\end{gathered} \tag{10}$$

where from prior definitions

$$D_2 = \begin{bmatrix}\beta^2 & 0 \\ 0 & \alpha^2\end{bmatrix} \quad , \quad D_2^{-1} = \frac{}{\alpha^2\beta^2}\begin{bmatrix}\alpha^2 & 0 \\ 0 & \beta^2\end{bmatrix} = B_1\,. \tag{11}$$

It follows that

$$\left[\frac{\left(ik_z\right)}{\left(-i\omega\right)}I + B_1\right]\tilde{\mathbf{u}} = 0 \quad \rightarrow \quad \mathbf{u}_z + B_1\mathbf{u}_t = 0 \tag{12}$$

and then to

$$\mathbf{u}_{zt} + B_1\mathbf{u}_{tt} = 0 \tag{13}$$

with

$$B_1 = C_1 = \begin{bmatrix}1/\beta & 0 \\ 0 & 1/\alpha\end{bmatrix}. \tag{14}$$

<u>Second order paraxial approximation:</u>

Following Clayton and Engquist (1977) the following higher order paraxial equation may be written

$$\mathbf{u}_{tz} + C_1\mathbf{u}_{tt} + C_2\mathbf{u}_{tx} + C_3\mathbf{u}_{xx} = 0 \tag{15}$$

It will be required to determine the matrices C_2 and C_3. The Fourier transform of (15) may be found to be

$$\left[I\frac{\left(ik_z\right)}{\left(-i\omega\right)} = -C_1 - C_2\frac{\left(ik_x\right)}{\left(-i\omega\right)} - C_3\frac{\left(ik_x\right)^2}{\left(-i\omega\right)^2}\right]\tilde{\mathbf{u}} = 0 \quad \left(\tilde{\mathbf{u}} \neq 0\right). \tag{16}$$

The coefficients of the coupled wave equations are required to satisfy

$$\left[I - D_2 \frac{(ik_z)^2}{(-i\omega)^2} - H \frac{(ik_x)(ik_z)}{(-i\omega)^2} - D_1 \frac{(ik_x)^2}{(-i\omega)^2} \right] = 0 \tag{17}$$

Substitute equation (16) into equation (17).

$$\begin{aligned} &\left[I - D_2 \left(C_1 + C_2 \frac{(ik_x)}{(-i\omega)} + C_3 \frac{(ik_x)^2}{(-i\omega)^2} \right) \left(C_1 + C_2 \frac{(ik_x)}{(-i\omega)} + C_3 \frac{(ik_x)^2}{(-i\omega)^2} \right) + \right. \\ &\left. H \frac{(ik_x)}{(-i\omega)} \left(C_1 + C_2 \frac{(ik_x)}{(-i\omega)} + C_3 \frac{(ik_x)^2}{(-i\omega)^2} \right) - D_1 \frac{(ik_x)^2}{(-i\omega)^2} \right] = 0 \end{aligned} \tag{18}$$

Expanding and retaining only terms in $\left(ik_x / -i\omega \right)$ results in

$$-D_2 C_1 C_2 - D_2 C_2 C_1 + H C_1 = 0 \tag{19}$$

plus higher order terms in powers of $\left(ik_x / -i\omega \right)$ and noticing that from the zero approximation

$$I - D_2 (C_1)^2 = 0 \quad \left[B_1 = C_1 = \begin{bmatrix} 1/\beta & 0 \\ 0 & 1/\alpha \end{bmatrix} \right] \tag{20}$$

It is not difficult to argue that C_2 should be of the form

$$C_2 = \begin{bmatrix} 0 & C_2^{(12)} \\ C_2^{(21)} & 0 \end{bmatrix} \tag{21}$$

and that higher order terms in powers of $\left(ik_x / -i\omega \right)$ should be of no consequence and will be ignored.

After a number of matrix multiplications, which have been deleted here, the following sequence of derivations are required to be made.

$$C_2^{(12)} \left(\beta + \beta^2 / \alpha \right) - \left(\alpha^2 - \beta^2 \right) / \alpha = 0 \tag{22}$$

$$\begin{aligned} C_2^{(12)} \left(\frac{\alpha\beta + \beta^2}{\alpha} \right) &= \frac{\left(\alpha^2 - \beta^2 \right)}{\alpha} \\ C_2^{(12)} = \frac{\left(\alpha^2 - \beta^2 \right)}{\alpha} \frac{\alpha}{\beta(\alpha + \beta)} &= \frac{(\alpha + \beta)(\alpha - \beta)}{\beta(\alpha + \beta)} \end{aligned} \tag{23}$$

One required quantity is found as:

$$C_2^{(12)} = \frac{(\alpha - \beta)}{\beta} \tag{24}$$

The second term is obtained from the following sequence,

$$-\alpha C_2^{(21)} - \frac{\alpha^2 C_2^{(21)}}{\beta} + \frac{(\alpha^2 - \alpha^2)}{\beta} = 0 \tag{25}$$

$$C_2^{(21)}\left(\frac{\alpha\beta + \alpha^2}{\beta}\right) = \frac{(\alpha^2 - \beta^2)}{\beta} = \frac{(\alpha - \beta)(\alpha + \beta)}{\beta} \tag{26}$$

$$C_2^{(21)} = \frac{\beta(\alpha - \beta)(\alpha + \beta)}{\alpha\beta(\alpha + \beta)} \tag{27}$$

$$C_2^{(21)} = \frac{(\alpha - \beta)}{\alpha}, \tag{28}$$

so that the matrix is given by

$$C_2 = \begin{bmatrix} 0 & C_2^{(12)} \\ C_2^{(21)} & 0 \end{bmatrix} = (\alpha - \beta)\begin{bmatrix} 0 & 1/\beta \\ 1/\alpha & 0 \end{bmatrix}. \tag{29}$$

The following quantity is required later in the derivation of the terms in C_3.

$$(C_2)^2 = (\alpha - \beta)^2 \begin{bmatrix} 0 & 1/\beta \\ 1/\alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & 1/\beta \\ 1/\alpha & 0 \end{bmatrix} = (\alpha - \beta)^2 \begin{bmatrix} 1/\alpha\beta & 0 \\ 0 & 1/\alpha\beta \end{bmatrix} \tag{30}$$

Finally it is required to determine C_3. As with C_2 the starting equation is (18) Expand this equation and retaining only those terms in powers of $(ik_x)^2/(-i\omega)^2$ and equating this quantity to zero has

$$-D_2C_1C_3 - D_2C_2C_2 - D_2C_3C_1 + HC_2 - D_1 = 0\,. \tag{31}$$

From observation, C_3 should be of the form:

$$C_3 = \begin{bmatrix} C_3^{(11)} & 0 \\ 0 & C_3^{(22)} \end{bmatrix} \tag{32}$$

After all the matrix multiplications in (31) and collecting terms the following sequence produces one of the desired results,

$$-\beta C_3^{(11)}-\beta C_3^{(11)}-(\alpha-\beta)^2\frac{\beta}{\alpha}+\frac{\left(\alpha^2-\beta^2\right)(\alpha-\beta)}{\alpha}-\alpha^2=0 \tag{33}$$

$$-2\beta C_3^{(11)}-\frac{(\alpha-\beta)^2}{\alpha}\left(\beta-(\alpha+\beta)\right)-\alpha^2=0 \tag{34}$$

$$\begin{aligned}&-2\beta C_3^{(11)}+(\alpha-\beta)^2-\alpha^2=0\\&-2\beta C_3^{(11)}+\alpha^2-2\alpha\beta+\beta^2-\alpha^2=0\\&-2\beta C_3^{(11)}-(2\alpha-\beta)=0\end{aligned} \tag{35}$$

which is:

$$C_3^{(11)}=\frac{(\beta-2\alpha)}{2} \tag{36}$$

The second term is obtained in the a similar manner

$$-\alpha C_3^{(22)}-\alpha C_3^{(22)}-\frac{\alpha(\alpha-\beta)^2}{\beta}+\frac{\left(\alpha^2-\beta^2\right)(\alpha-\beta)}{\beta}-\beta^2=0 \tag{37}$$

$$-2\alpha C_3^{(22)}-\frac{\alpha(\alpha-\beta)^2}{\beta}+\frac{\left(\alpha^2-\beta^2\right)(\alpha-\beta)}{\beta}-\beta^2=0 \tag{38}$$

$$-2\alpha C_3^{(22)}-\frac{(\alpha-\beta)^2}{\beta}\left(\alpha-(\alpha+\beta)\right)-\beta^2=0 \tag{39}$$

$$\begin{aligned}&-2\alpha C_3^{(22)}+(\alpha-\beta)^2-\beta^2=0\\&-2\alpha C_3^{(22)}+\alpha^2-2\alpha\beta+\beta^2-\beta^2=0\\&-2C_3^{(22)}+\alpha-2\beta=0\end{aligned} \tag{40}$$

$$C_3^{(22)}=\frac{\alpha-2\beta}{2} \tag{41}$$

This finally results in the following

$$C_3=\frac{1}{2}\begin{bmatrix}(\beta-2\alpha) & 0\\ 0 & (\alpha-2\beta)\end{bmatrix}. \tag{42}$$

NUMERICAL RESULTS

Given the velocity/density versus depth structure shown in Fig.1, VSP synthetics were computed for the cases with no damping at the model bottom, Figs. 2 and 4 for the vertical and radial components of displacement. The synthetic traces shown in Figs. 3 and 5 employ the damping derived here, which appears in the paper of Clayton and Engquist (1977). The finite difference analogues appear in the Appendix of that paper. There is one minor typographic error in that Appendix, which is left for the reader to determine.

CONCLUSIONS

The paper of Clayton and Engquist (1977) is revisited and some clarifications of the method employed there to construct paraxial wave equations approximations. The minor explanatory derivations presented here may assist those who wish to use a similar method for more complex media types.

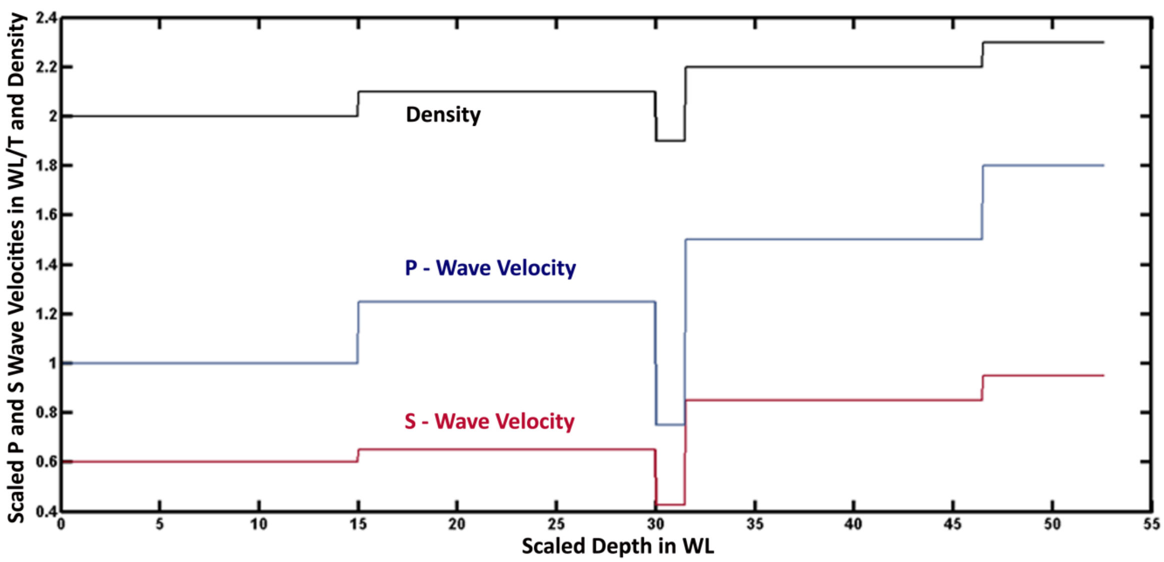

Fig. 1: The scaled velocity/density depth model used in the computation of the synthetic VSP traces computed in this report.

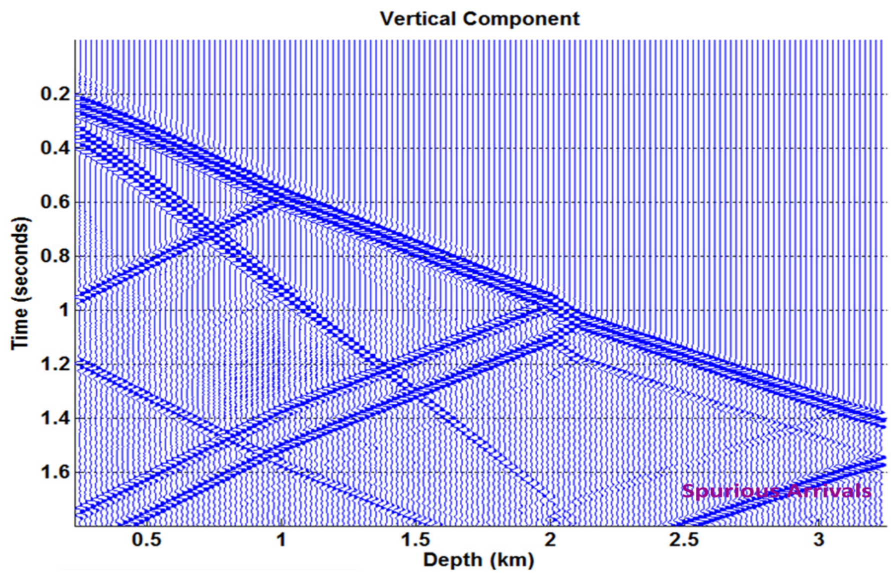

Fig. 2: The vertical component of the VSP synthetic of the model described in the text. In this panel the computations continued to the proper time indicating that *spurious reflections* from the model bottom are included in the traces.

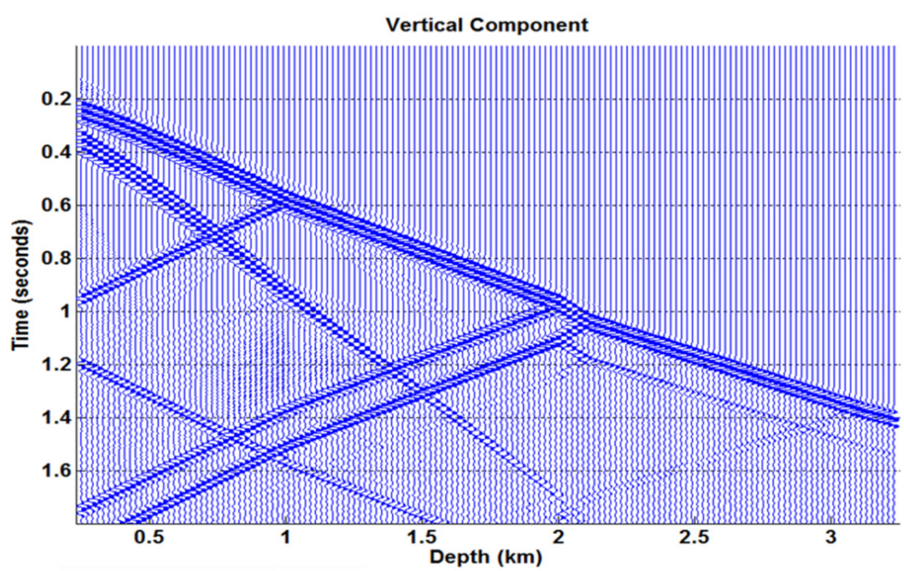

Fig. 3: The vertical component of the VSP synthetic of the model described in the text. In this panel the computations continued to the proper time indicating that no spurious reflections from the model are included in the traces.

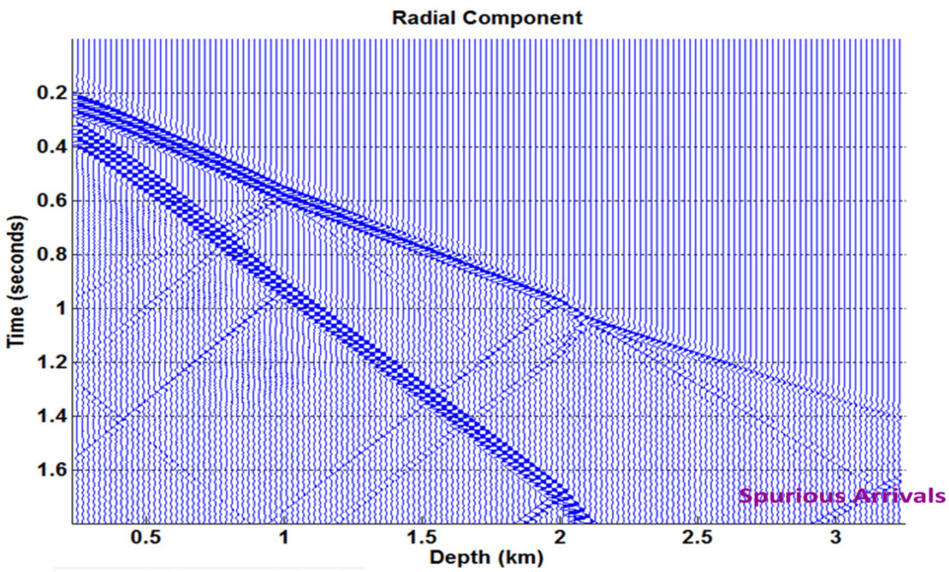

Fig. 3: The radial component of the VSP synthetic of the model described in the text. In this panel the computations continued to the proper time indicating that spurious reflections from the model bottom are included in the traces

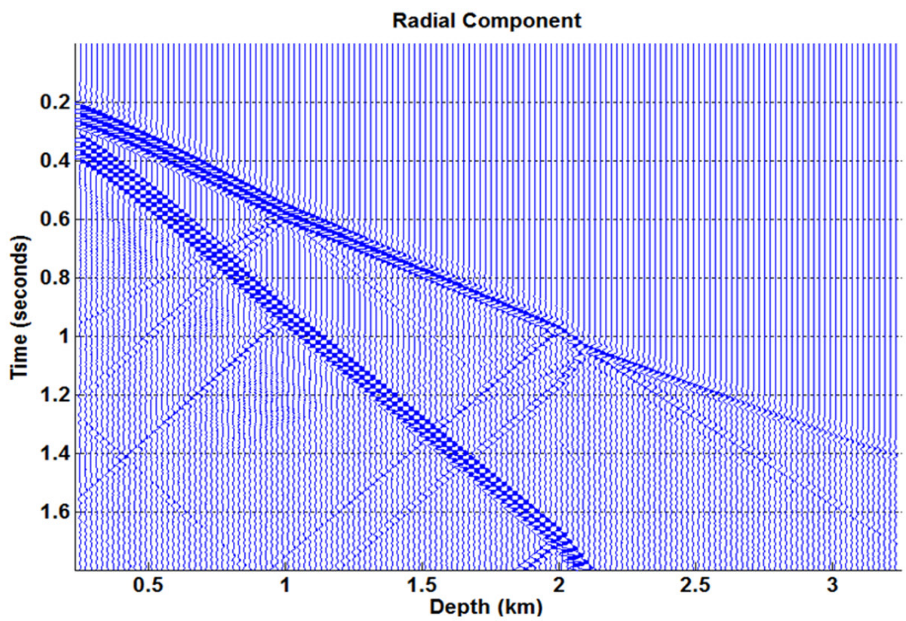

Fig. 4: The radial component of the VSP synthetic of the model described in the text. In this panel the computations continued to the proper time indicating that no spurious reflections from the model are included in the traces

REFERENCES

Ames, W.F., 1969, Numerical Methods for Partial Differential Equations: Barnes & Noble, New York.

Cerjan, C., Kosloff, D., Kosloff, R. and Reshef, M., 1985, A nonreflecting boundary condition for discrete acoustic and elastic wave equations, Geophysics, 50, 705-708.

Clayton, R. and Enquist, B., 1977, Absorbing boundary conditions for acoustic and elastic wave equations, Bulletin of the Seismological Society of America, 67, 1529-1540.

Gazdag, J., 1973, Numerical convective schemes based on the accurate computation of space derivatives, Journal of Computational Physics, 13, 100-113.

Gazdag, J., 1981, Modeling of the acoustic wave equation with transform methods, Geophysics, 46, 854-859.

Kosloff, D. and Baysal, E., 1982, Forward modeling by a Fourier method, Geophysics, 47, 1402-1412.

Krebes, E.S. and Daley, P.F., MMMM,

Mitchell, A.R., 1969, Computation Methods in Partial Differential Equations: John Wiley & Sons, Inc.

Reynolds, A.C., 1978, Boundary conditions for the numerical solution of wave propagation problems, Geophysics, 43, 1099-1110.

Richtmeyer, R.D. and Morton, K.W., 1967, Difference Methods for Initial-value Problems, Krieger Publishing Company.

Smarskii, A. A., 1971, Introduction to the Theory of Difference Schemes, Nauka, Moscow (in Russian).

Sneddon, I.A, 1995, Fourier Transforms, Dover Publications, New York.

Zahradnik, J., P. Moczo, and F. Hron, 1993, Testing four elastic finite difference schemes for behavior at discontinuities, Bulletin of the Seismological Society of America, 83, 107-129.

Zhu, X. and McMechan, G.A., 1991. Finite-difference modeling of the seismic response of fluid saturated, porous, elastic media using Biot theory. Geophysics, 56, 424-435.