PP and PS reflection and transmission coefficients in anelastic media: the anomalous case

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ABSTRACT

Using the derivation of reflection and transmission coefficients at the interface between two elastic given in Aki and Richards (1980) as a starting point, modified coefficients may be obtained for the case of two anelastic media in welded contact at a similar (plane) boundary. The anelastic equivalents of two elastic velocity distributions are considered for $P-$wave incidence on the plane boundary separating the two medium. Figures showing the plots of amplitude and phase of the four coefficients P_1P_1, P_1P_2, P_1S_1 and P_1S_2 are produced with the results from the elastic case included in all of the plots. As the term "anomalous" requires some preliminary introduction of terms, its definition, used in context here, will be left be left until later in the report.

INTRODUCTION

The topic of wave propagation in anelastic media has been an area of study and moderate controversy for about a century. There are not many matters on this subject that have been resolved to the satisfaction of all. The second author listed above has written an extremely comprehensive paper in which a significant amount of the related literature is cited (Krebes and Daley, 2007). Consequently, they will not be repeated here in their entirety. The first author's views, tempered by the second author, are presented in the papers, Daley and Krebes (2004) and Daley and Krebes (2015). What is hoped is that what can be achieved here is that the anelastic reflection and transmission coefficients presented here *appear* as one would expect them to realistically look like. For this purpose two models have been chosen for investigation. The first is a very standard model often seen in actual field data (at least where the velocity of the elastic waves are concerned). This *elastic* model is given by the inequality, $\alpha_2 > \alpha_1 > \beta_2 > \beta_1$, is shown schematically in Fig.1.and defined in Table 1. Here, α and β are the real values of the compressional P and shear S velocities in an *elastic* media, with the subscripts indicating the medium, $1-$upper medium and $2-$lower medium (Fig. V). The values of the quality factors for the four modes of propagation also appear in Table 1.and are inferred in Fig.1. The inequality for these is $\left(Q_P^{(2)}, Q_S^{(2)}, Q_S^{(1)}\right) > Q_P^{(1)}$.

The second model considered is one not often encountered in field data, but is a possibility. As indicated in Fig.2. it is marginally more complex. The elastic model for this is given by $\alpha_2 > \beta_2 > \alpha_1 > \beta_1$ with the same quality factors used for model 1.

Agreement may be reached for the figures produced for model 1. Model 2 introduces a number of other possible debatable points plus a consensus that further investigation is required.

THEORY

The incident horizontal component of the slowness vector (p) associated with an incident $P-$ wave lies along the straight line between the points $(0,0)$ and the point, p_1, in the first quadrant of the complex $p-$ plane, inclusive of the real axis. A specific location on this line will be denoted as p_0. In an elastic medium for $P-$ wave incidence, p_0 has the more familiar form, $p_0 = \sin\theta_1/\alpha_1$, a real quantity, related to all modes of propagation by Snell's Law. For the anelastic case, p_0 is in general complex and an analogue of Snell's Law is used.

$$\mathrm{Re}[p_1] = 1/\alpha_1 \qquad \mathrm{Re}[p_2] = 1/\alpha_2\,. \tag{1}$$

$$\mathrm{Re}[p_3] = 1/\beta_1 \qquad \mathrm{Re}[p_4] = 1/\beta_2\,. \tag{2}$$

Apart from the elastic parameters of the two media, α_j, β_j and ρ_j $(j=1,2)$, the $P-$ wave and $S-$ wave velocities and density in the two media, the factors $Q_P^{(j)}$ and $Q_S^{(j)}$ $(j=1,2)$ are assumed to be known. These control the amount of dissipation attributable to the waves associated with each of these modes in the medium of propagation.

Assuming that the quantities $Q_P^{(j)}$ and $Q_S^{(j)}$ $(j=1,2)$ are such that $Q_P^{(j)} >> 1$ and $Q_S^{(j)} >> 1$ then the following may be used. For numerical applications, it saves only minimal time and the exact expressions can be used. However, they are included here as many readers may be more familiar with the second equations for p_j $(j=1,4)$, the complex horizontal components of the slowness vector associated with each mode of propagation.

$$p_1^2 = \frac{1}{\alpha_1^2}\left(1+\frac{i}{Q_P^{(1)}}\right) \;\rightarrow\; p_1 \approx \frac{1}{\alpha_1}\left(1+\frac{i}{2Q_P^{(1)}}\right) \tag{3}$$

$$p_2^2 = \frac{1}{\alpha_2^2}\left(1+\frac{i}{Q_P^{(2)}}\right) \;\rightarrow\; p_2 \approx \frac{1}{\alpha_2}\left(1+\frac{i}{2Q_P^{(2)}}\right) \tag{4}$$

$$p_3^2 = \frac{1}{\beta_1^2}\left(1+\frac{i}{Q_S^{(1)}}\right) \;\rightarrow\; p_3 \approx \frac{1}{\beta_1}\left(1+\frac{i}{2Q_S^{(1)}}\right) \tag{5}$$

$$p_4^2 = \frac{1}{\beta_2^2}\left(1+\frac{i}{Q_S^{(2)}}\right) \;\rightarrow\; p_4 \approx \frac{1}{\beta_2}\left(1+\frac{i}{2Q_S^{(2)}}\right) \tag{6}$$

In Aki and Richards (1980), the vertical components of the slowness vector for the four modes of propagation (five if the incident wave is included) are given in the form of cosines as

$$\xi_j = \frac{\cos\theta_j}{\alpha_j} \qquad (j=1,2) \quad \text{for } P \text{ - waves}. \tag{7}$$

and

$$\eta_j = \frac{\cos\theta_j}{\beta_j} \qquad (j=1,2) \quad \text{for } S \text{ - waves}. \tag{8}$$

As the ξ_j and η_j are generally complex quantities in the anelastic case they are most often written as the radicals

$$\xi_j = \left(p_j^2 - p^2\right)^{1/2} \qquad (j=1,2). \tag{9}$$

$$\eta_j = \left(p_{j+2}^2 - p^2\right)^{1/2} \qquad (j=1,2). \tag{10}$$

where at the saddle point, $p \to p_0$. For p_0 near the branch point at p_2, ξ_2 may be approximated as

$$\begin{aligned} \xi_2 &= \left(p_2 + p_0\right)^{1/2}\left(p_2 - p_0\right)^{1/2} \\ \xi_2 &\approx \left(2p_2\right)^{1/2}\left(p_2 - p_0\right)^{1/2} \end{aligned}. \tag{11}$$

which is not used in this report but is worthy of note.

At this point a theoretical overview of this topic should be given. However, this has been done in another publication (Daley and Krebes, 2015). It is also not for general consumption as the contents refer on numerous occasions to a graduate complex variable theory text similar to Alfors (1969) as well as to a text on classical and quantum mechanics (Razavy, 2005).

The term *anomalous* has taken too long to be defined. Consider Model 1, defined in Table 1 and shown schematically in Figure 2. If $Q_P^{(2)}$ were less than $Q_P^{(1)}$, the branch point p_2 would lie above the saddle point progression path $(p = 0 \text{ to } p = p_1)$ and as a consequence, would not cross the branch cut corresponding to $p = p_2$. In this case there would be no need to take any type of corrective action to produce results similar to the related elastic case. In Figure 2 it is clear that the model parameters are such that the saddle point progression path crosses a branch cut, specifically the one associated with $p = p_2$. The formulae used to compute reflection and transmission coefficients in this case produce results not consistent with the elastic case. This has been termed the *anomalous* case. Without going into lengthy detail it required that the radical associated with $p = p_2$ $\left(\xi_2 = \left(p_2^2 - p_0^2\right)^{1/2}\right)$ is required to remain an analytic function (no discontinuous first derivative) in the first quadrant of the complex $p-$plane. A précised explanation of this is given in Daley and Krebes (2015). For more detailed analysis a graduate level complex variable analysis text should be consulted.

NUMERICAL RESULTS

What is considered here are, for lack of better terminology, will be referred to as the *bad* or *anomalous* cases of $P-$wave incidence from half space 1 (upper) on an interface between two anelastic medium, with medium 2 referring to the lower half space (Figure 1). The *bad* designation comes from the fact that $Q_P^{(1)}$ is less than either or both of the quantities $Q_P^{(2)}$ and $Q_S^{(2)}$. The saddle point progression line is a straight line between the origin and the point $p = p_1$ in the upper right quadrant of the complex horizontal slowness (p) plane. The result is that this line must cross either the branch cut associated with the $P-$wave in medium 2 (Figure 2) or both this branch cut and the one linked to the $S-$wave in medium (Figure 3).

Before proceeding further, it should be noted that the measures taken here to get what are perceived to be reasonable results are not required if $Q_P^{(1)} > Q_P^{(2)}$ and/or $Q_S^{(2)}$ (the *good* case). It was for this reason that this problem was addressed. It was difficult to believe that some infinitesimal change in one medium parameter could dramatically change the results of the reflection and transmission coefficients.

The first case to be considered is shown in Figure 2 where $Q_P^{(1)} < Q_P^{(2)}$. This is similar to the S_H discussed in (Daley and Krebes, 2015). The media parameters are given in Table 1 and the reflection and transmission coefficients for $P-$wave incidence from the upper half space 1 are presented in figures 4 – 9. Amplitude and phase are plotted for the four reflection and transmission coefficients, P_1P_1, P_1S_1, P_1P_2 and P_1S_2 versus the real part of p_1. The anelastic cases are shown in black, while the corresponding elastic cases are given in red. As a check, the real and imaginary parts of the coefficients P_1P_1 and P_1P_2 are presented in figures (5) and (8). In these two figures, the anelastic cases are shown in green and the elastic cases in red.

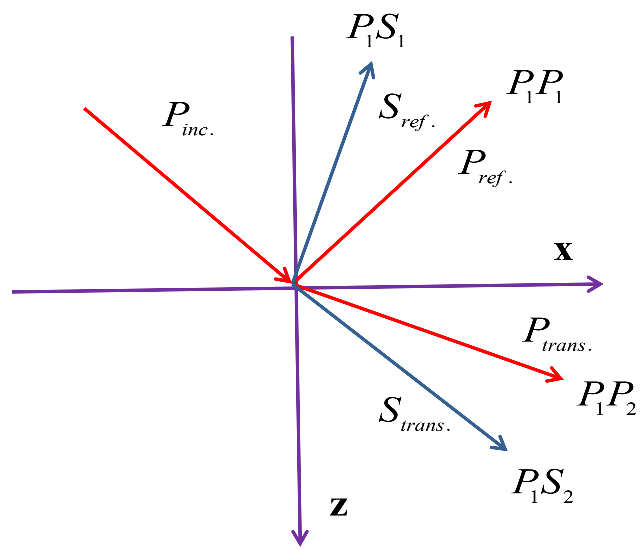

Fig.1. A schematic of a P – wave incident from the upper medium on an interface between two media and the resulting wave types that arise. The subscripts "*inc.*", "*ref.*" and "*trans.*" refer to incident, reflected and transmitted modes, respectively.

The reflection and transmission coefficients for Model 2 are displayed in figures 10 – 14. The positions of the branch points in the upper right quadrant of the complex $p-$plane are shown in Figure 3. The difference between Model 1 and Model 2 is clear. In Model 2 the saddle point progression path crosses two branch cut as it traverses the path from $p=0$ to $p=p_1$. The remainder of Model 2 is the same as Model 1 and the parameters used are given in Table 2. This is not a media type that is mathematically possible but not often or ever physically seen. However, it will be pursued here and its existence will be left for future discussion.

As for Model 1, the four reflection and transmission coefficients shown here are due to a $P-$wave incidence from the upper. The resulting coefficients are shown schematically in Figure 1 and include P_1P_1, P_1S_1, P_1P_2 and P_1S_2. As in the previous case, the amplitudes and phases of the generally complex coefficients are plotted versus the real the real part of p_1 (in black) with the elastic cases plotted in red, in Figures 10, 11, 13 and 14. Figure 12 is a plot of the real and imaginary parts of the P_1P_1 coefficient plotted against the real part of p_1 (in green) with the elastic case in red. The uncorrected (anomalous) case is not shown in the second set of figures related to Model 2 so as not overly clutter the figures.

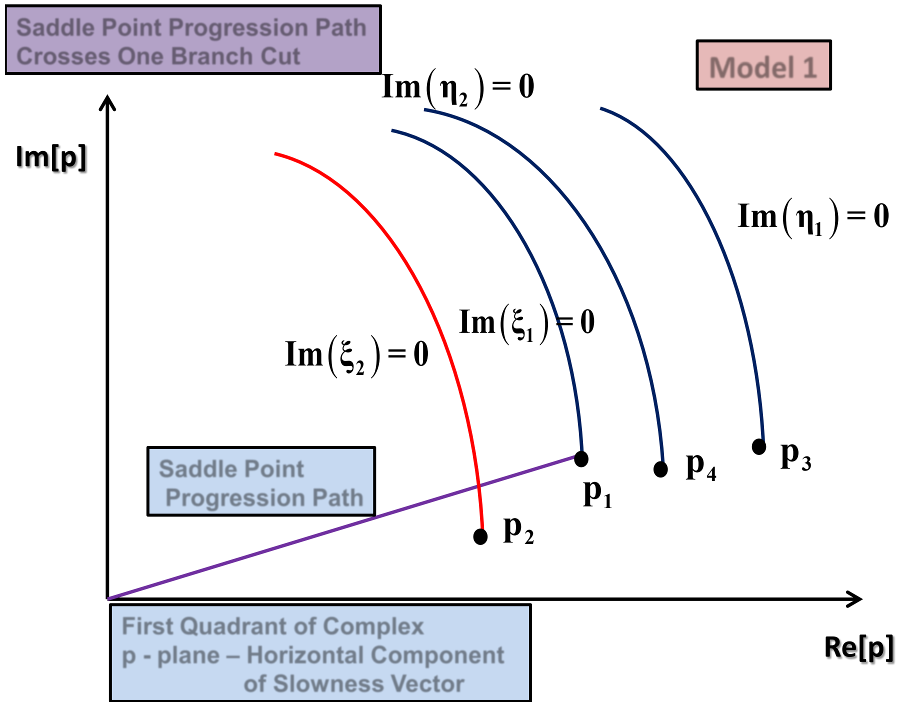

Fig.2. A schematic of the first quadrant of the complex *p* – plane (horizontal component of the slowness vector). The velocity model shown is a very standard one described in the text. The points p_j, $(j=1,4)$ are branch points of ξ_j and η_j, $(j=1,2)$ (vertical components of the related slowness vectors). The branch cuts for these are chosen to be the paths from p_j to ∞, such that $\mathrm{Im}[\xi_j]$ or $\mathrm{Im}[\eta_j]$ are equal to zero along the cut. The *saddle point progression path* is the values that the incident *P* – wave may have corresponding to the elastic case of $[0 \le \sin\theta/\alpha_1 \le 1;\ 0 \le \theta \le \pi/2]$. Of note in this figure is that the *saddle point progression path* crosses one branch cut, corresponding to the *P* – wave in the second medium.

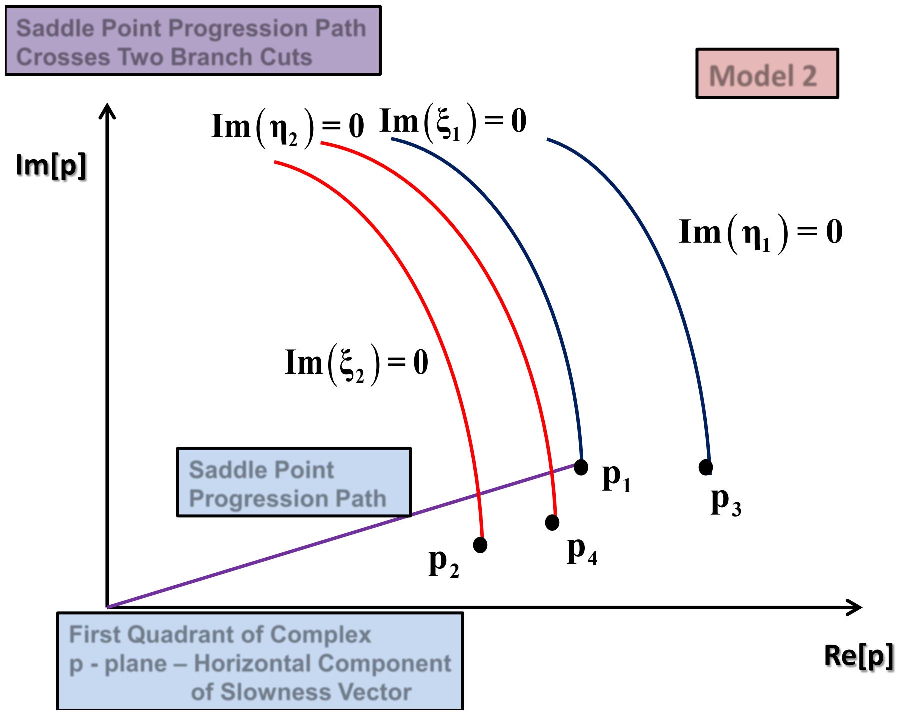

Fig.3. A schematic of the first quadrant of the complex *p* – plane (horizontal component of the slowness vector). The velocity model shown is a very standard one described in the text. The points p_j, $(j=1,4)$ are branch points of ξ_j and η_j, $(j=1,2)$ (vertical components of the related slowness vectors). The branch cuts for these are chosen to be the paths from p_j to ∞, such that $\text{Im}\left[\xi_j\right]$ or $\text{Im}\left[\eta_j\right]$ are equal to zero along the cut. The *saddle point progression path* is the values that the incident *P* – wave may have corresponding to the elastic case of $\left[0 \le \sin\theta/\alpha_1 \le 1;\ 0 \le \theta \le \pi/2\right]$. Of note in this figure is that the *saddle point progression path* crosses two branch cuts, associated with the *P* – wave and *S* – wave in the second medium.

Table 1. Medium parameters for Model 1. The real valued velocities $(\alpha \text{ and } \beta)$ have dimensions of km/s, density has the dimensions of gm/cm^3 and the Q_k $(k=P,S)$ are dimensionless.

Layer	α	β	ρ	Q_P	Q_S
1	1.0	0.57	1.0	10.0	20.0
2	1.5	0.82	1.0	15.0	22.0

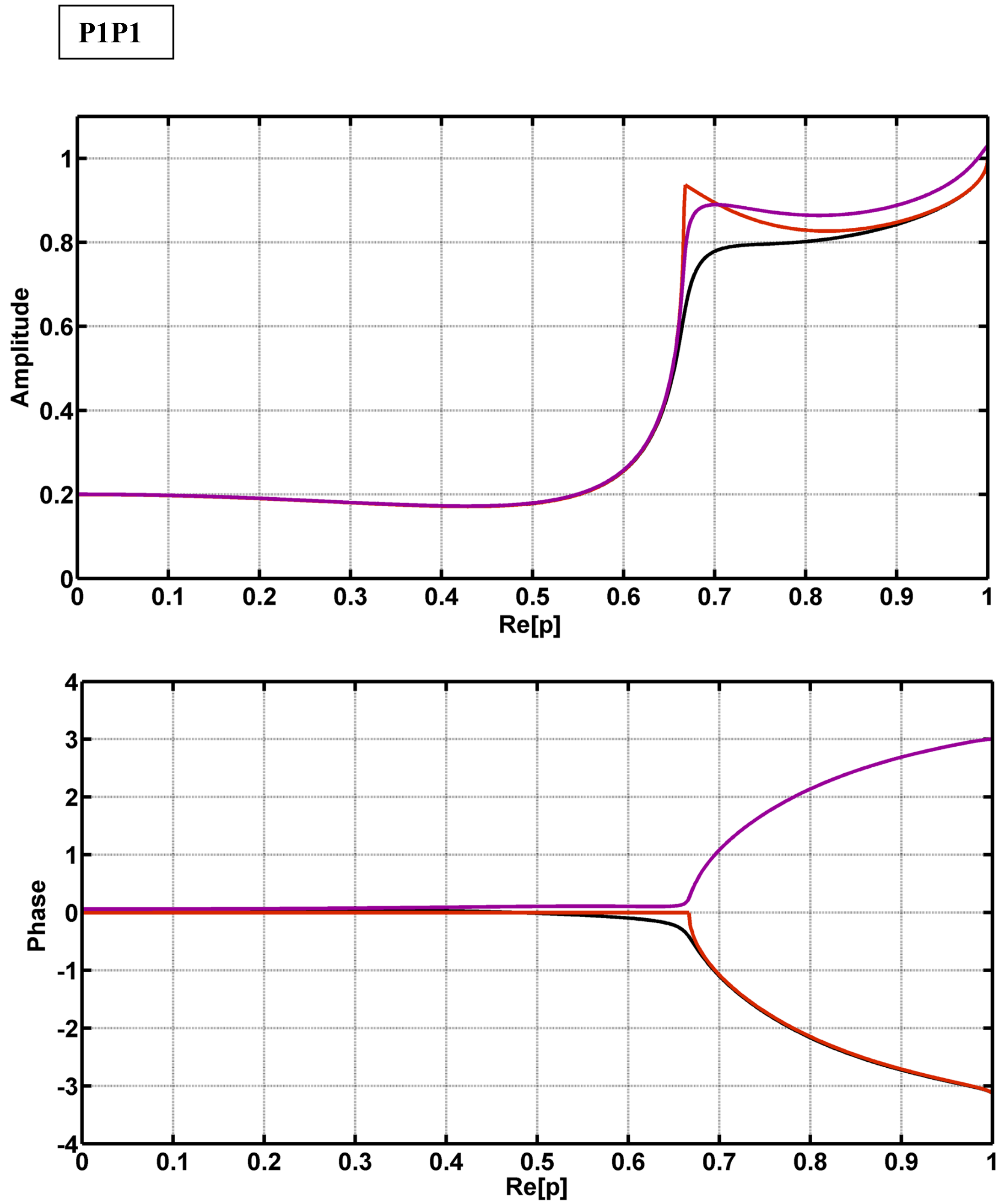

Fig.4. P_1P_1 reflection coefficient for Model 1 for P – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \le \mathrm{Re}[p_1] \le 1/\alpha_1\right)$. (Purple – the anelastic case before corrective action taken.)

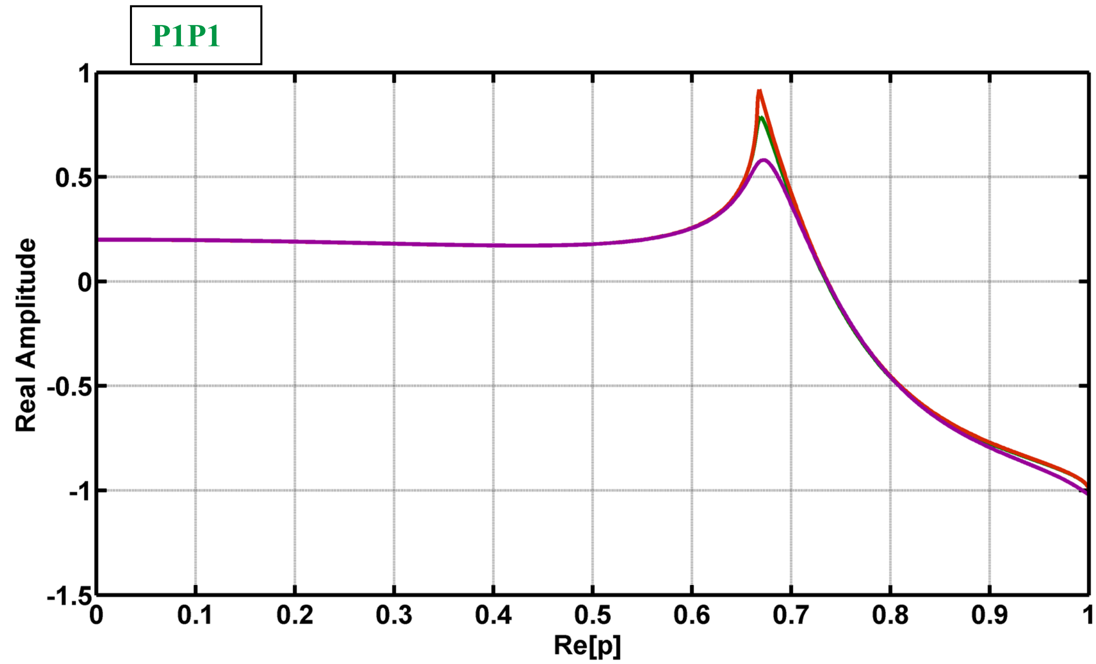

Fig.5. ***P_1P_1*** reflection coefficient for Model 1 for *P* – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Real and imaginary parts of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \le \mathrm{Re}[p_1] \le 1/\alpha_1\right)$. (Green – anelastic case.) (Purple – the anelastic case before corrective action taken.)

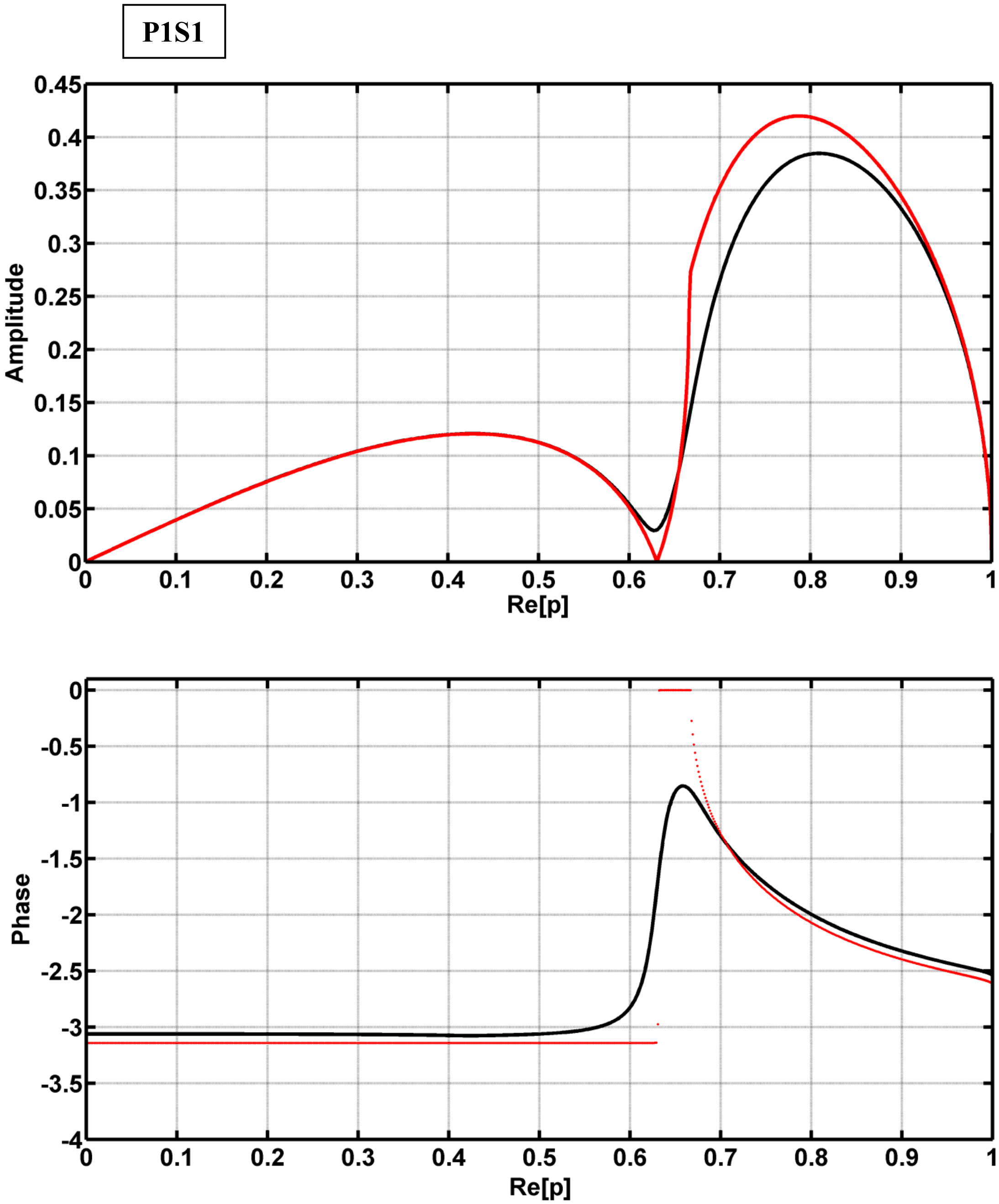

Fig. 6. P_1S_1 reflection coefficient for Model 1 for P – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \leq \mathrm{Re}[p_1] \leq 1/\alpha_1\right)$.

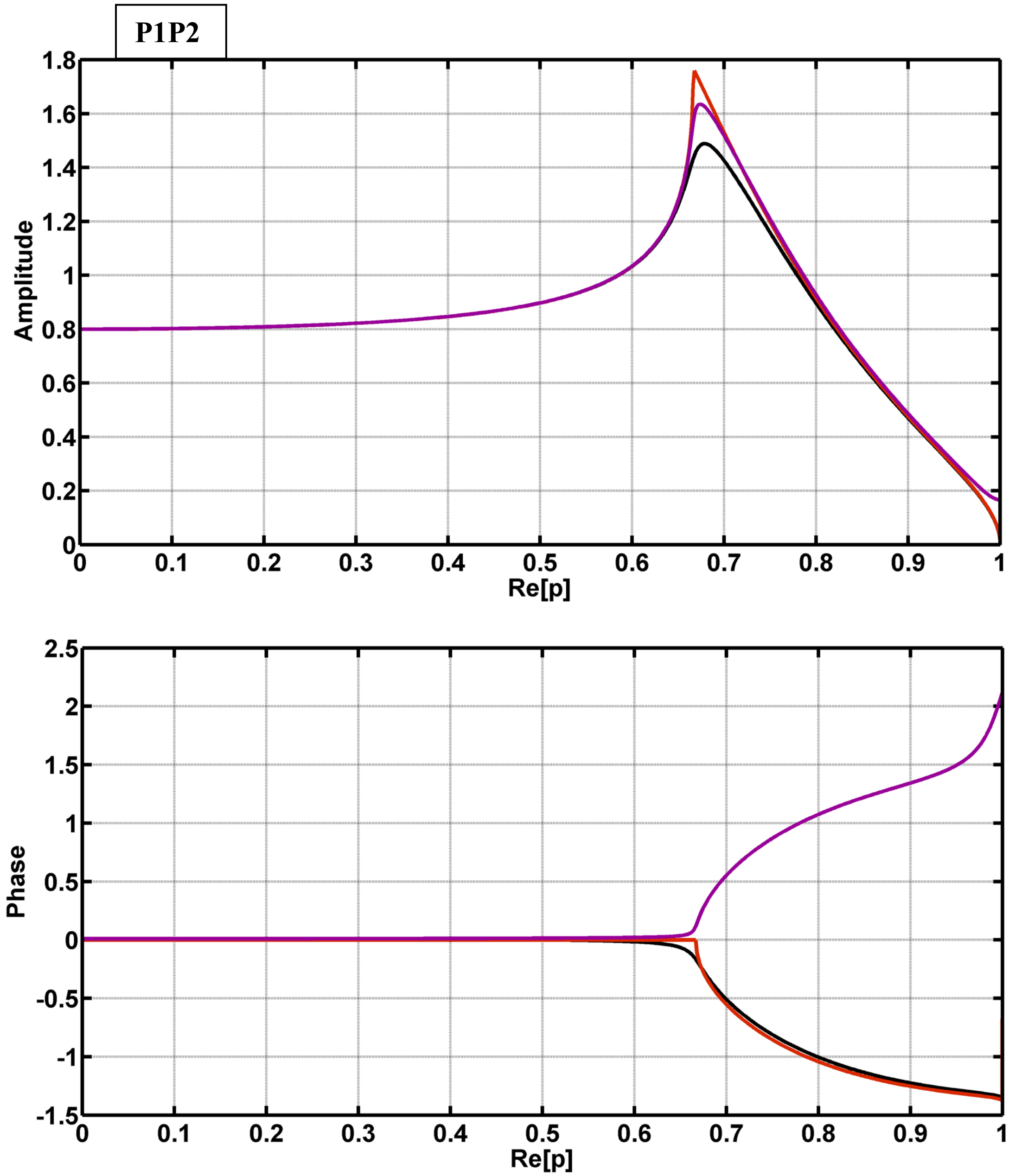

Fig.7. P_1P_2 transmission coefficient for Model 1 for P – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued transmission coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \le \mathrm{Re}[p_1] \le 1/\alpha_1\right)$. (Purple – the anelastic case before corrective action taken.)

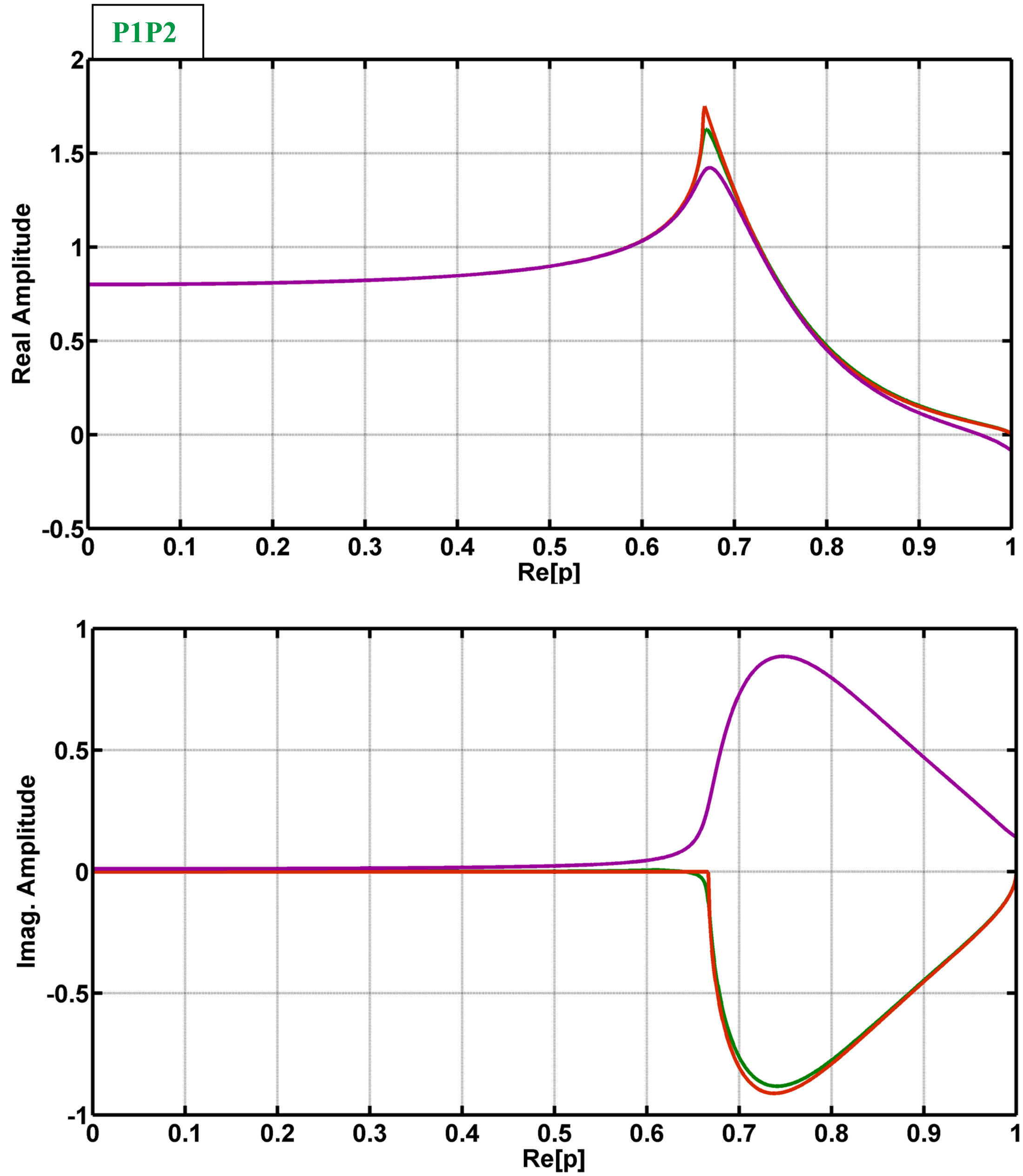

Fig.8. ***P_1P_2*** transmission coefficient for Model 2 for *P* – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Real and imaginary parts of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \le \mathrm{Re}[p_1] \le 1/\alpha_1\right)$. (Green – anelastic case.) (Purple – the anelastic case before corrective action taken.)

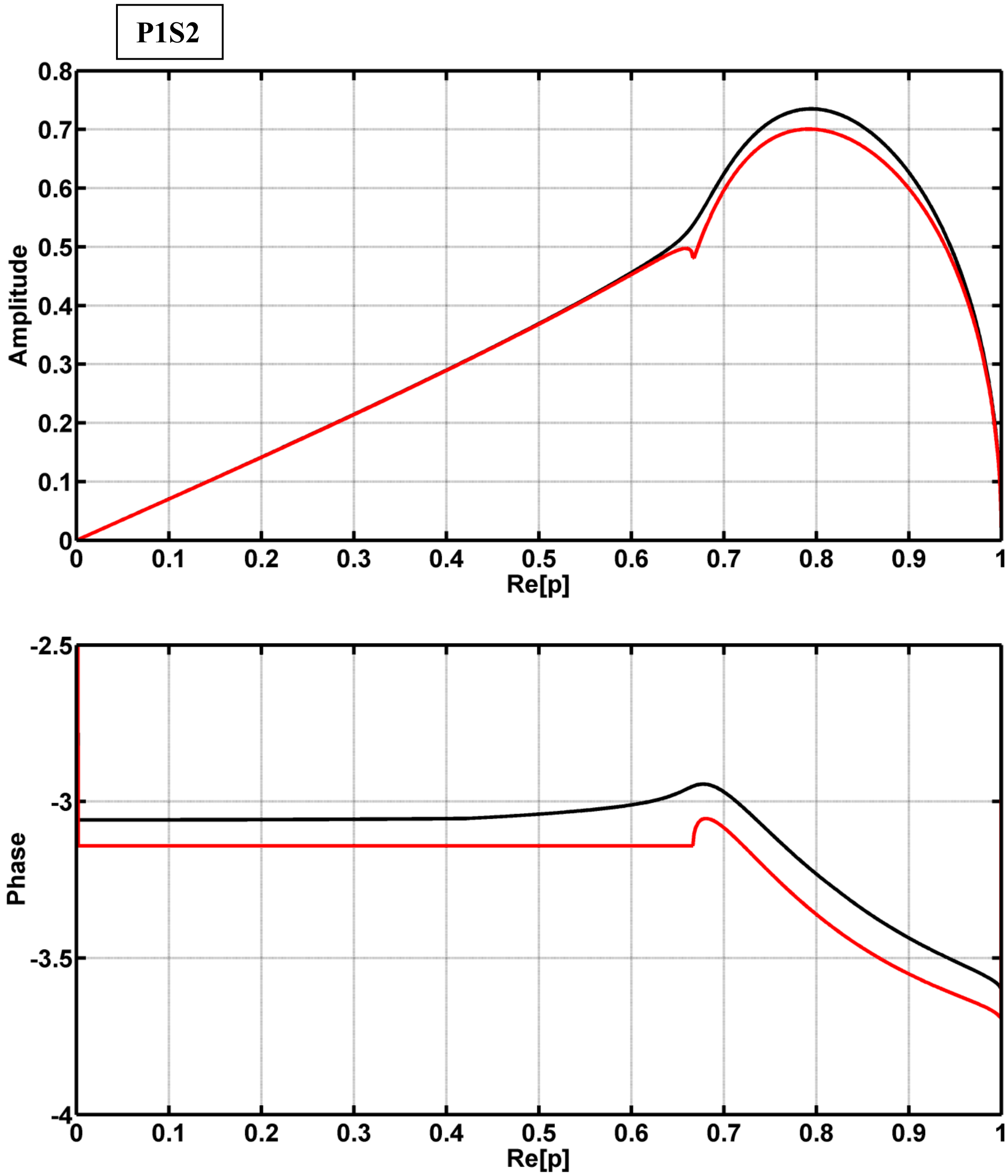

Fig.9. P_1S_2 transmission coefficient for Model 1 for P – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued transmission coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \le \mathrm{Re}[p_1] \le 1/\alpha_1\right)$.

Table 2. Medium parameters for Model 2. As in Model 1, the real valued velocities $(\alpha \text{ and } \beta)$ have dimensions of km/s, density has the dimensions of gm/cm^3 and the $Q_k \ (k = P, S)$ are dimensionless.

Layer	α	β	ρ	Q_P	Q_S
1	1.0	0.57	1.0	10.0	20.0
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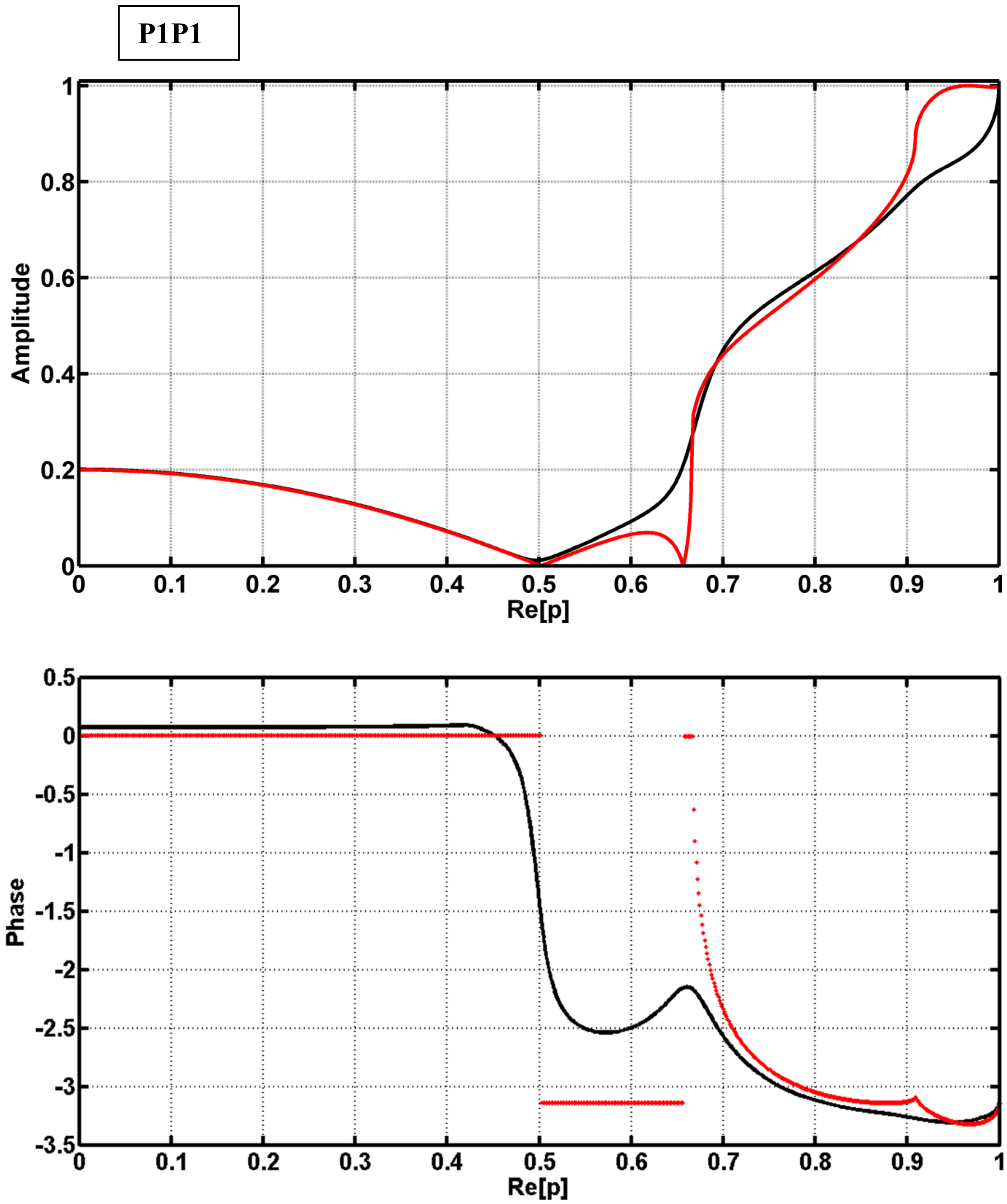

Fig.10. P_1P_1 reflection coefficient for Model 2 for P – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \le \mathrm{Re}[p_1] \le 1/\alpha_1\right)$

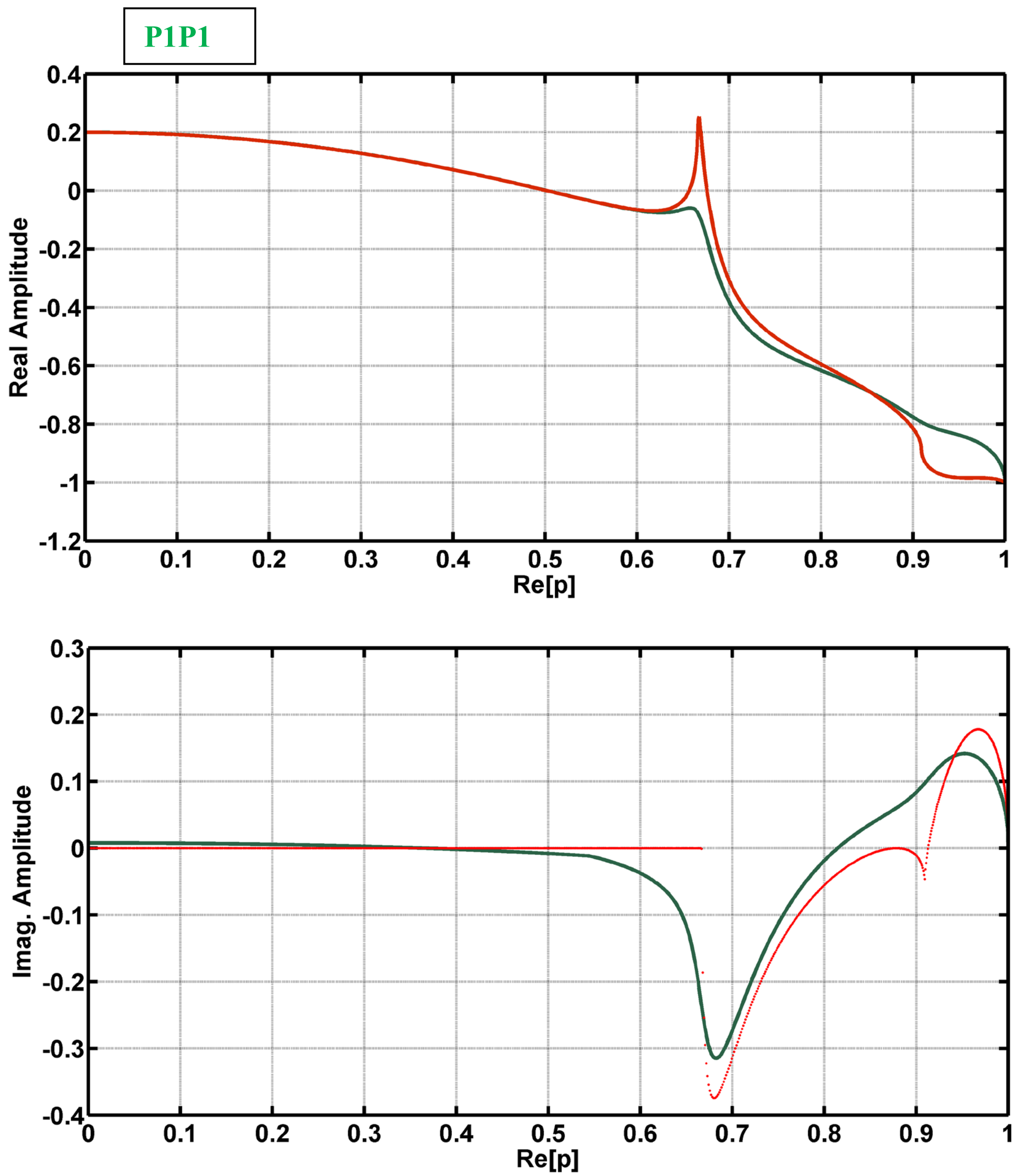

Fig.11. ***P_1P_1*** reflection coefficient for Model 2 for *P* – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \leq \mathrm{Re}[p_1] \leq 1/\alpha_1\right)$.

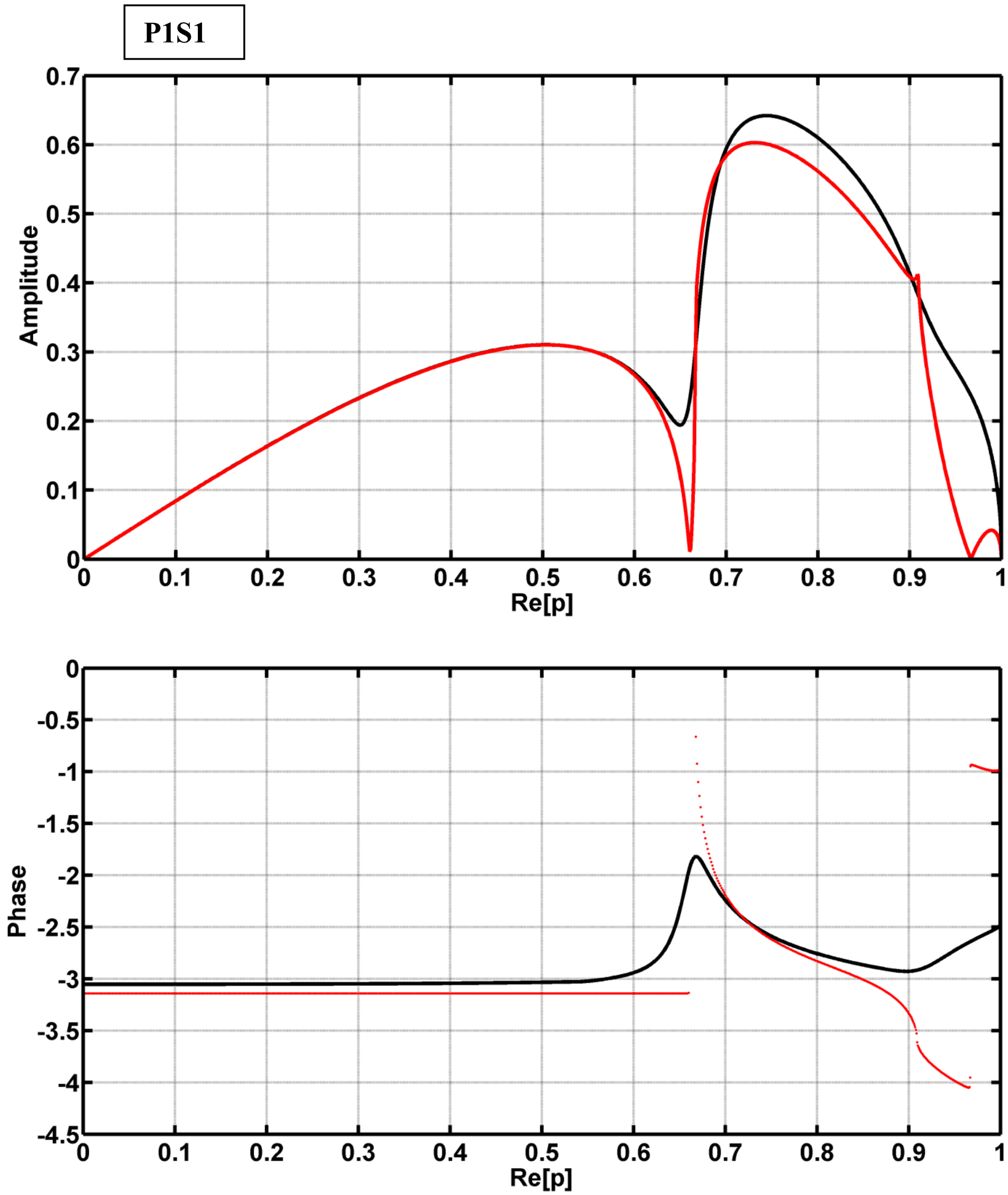

Fig.12. P_1S_1 reflection coefficient for Model 2 for P – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \le \mathrm{Re}[p_1] \le 1/\alpha_1\right)$.

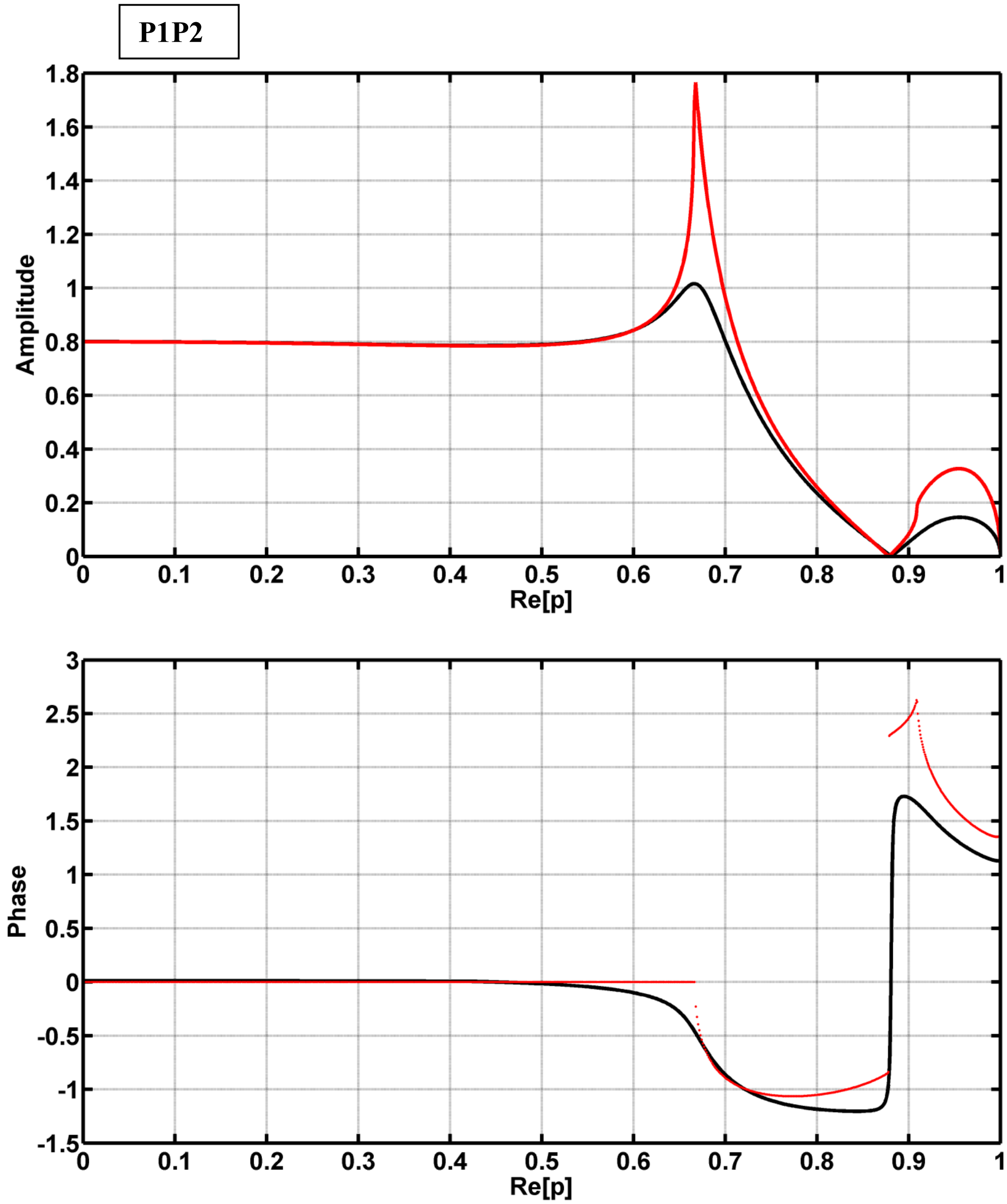

Fig.13. P_1P_2 reflection coefficient for Model 2 for P – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \le \mathrm{Re}[p_1] \le 1/\alpha_1\right)$.

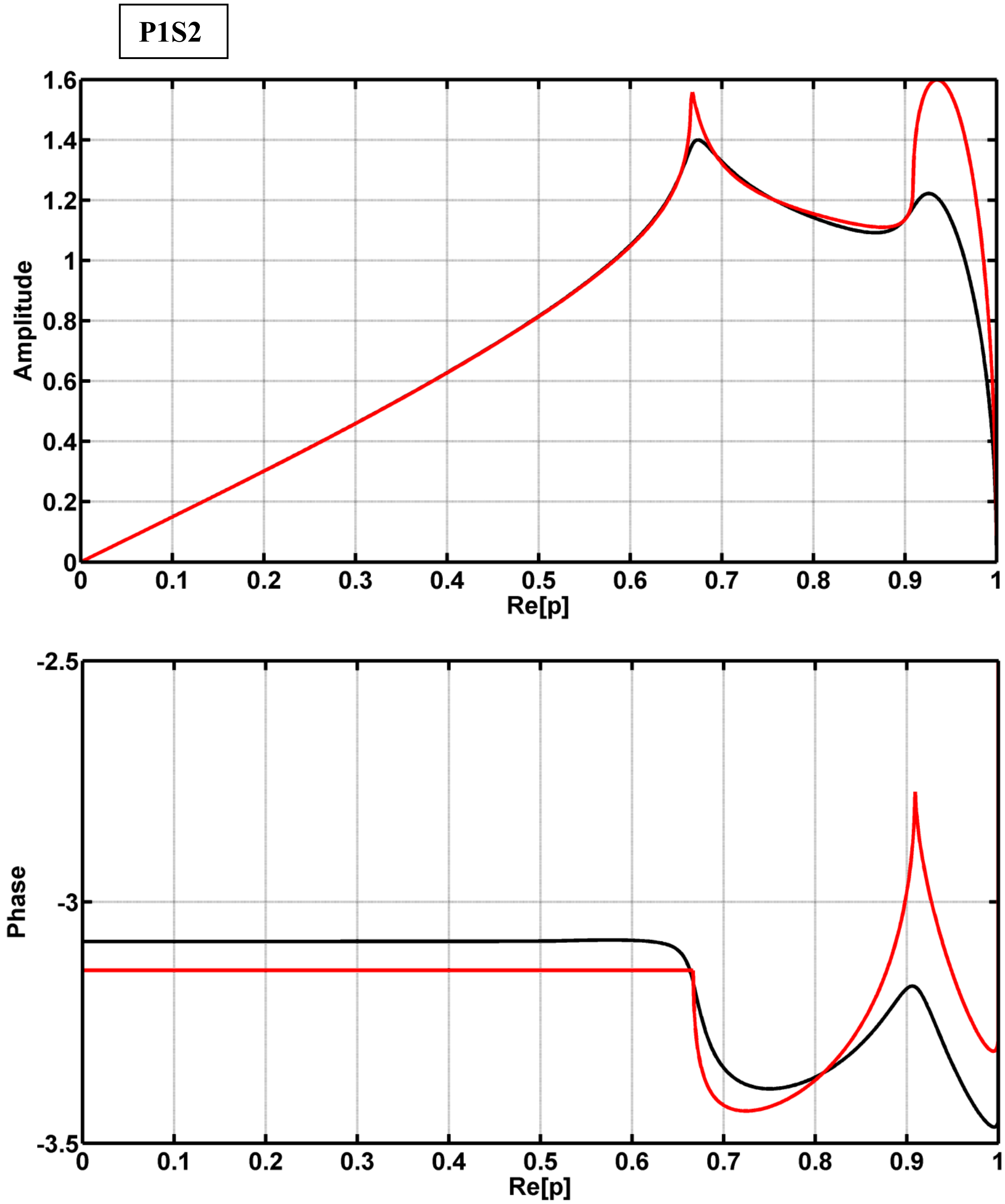

Fig.14. *P1S2* reflection coefficient for Model 2 for *P* – wave incidence from the upper medium onto the plane interface separating the upper and lower media. Amplitude and phase of the generally complex valued reflection coefficient are plotted versus the real part of the horizontal slowness vector $\left(0 \leq \mathrm{Re}[p_1] \leq 1/\alpha_1\right)$.

SUMMARY AND CONCLUSIONS

Reflection and coefficients for a plane $P-$wave incidence at the interface of a plane interface separating two anelastic isotropic models have been presented for two different models. The first set is for what has been termed the anomalous case, Krebes and Daley (2007). When compared to the elastic case, these coefficients appear to behave in manner which would be expected.

The second model used was similar to the first with only a minor modification of medium parameters. The results appear to be at least marginally consistent with the elastic case. However, there are some contentious points which warrant further investigation of this problem. These could be a consequence of a number of things, one being the nearness of two branch cuts to grazing incidence at $p_0 \approx p_1$.

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