Accuracy of numerical solutions to the elastic wave equation in multiple dimensions

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ABSTRACT

In this paper, we will solve for the exact solutions for reflection coefficients of the elastic wave equation in 1D, 2D, and 3D. The velocities in which we are most interested have a transition zone, or a portion of the velocity which is non-constant. As such, we will discuss what occurs at the start and end of the transition zone. In particular, we will find that certain continuity conditions are required. We will also discuss the case when the plane wave is orthogonal to the transition zone as well as the non-normal incidence case in 2D and 3D. This work is a extension of a paper by Hardeman and Lamoureux written in 2016. Finally, using the general solution for the reflection coefficients we find in the 1D and 2D cases, we will compare the results of the exact solution to numerical solutions of the elastic wave equation in 1D and 2D.

INTRODUCTION

Real world problems in seismic imaging are often difficult to solve analytically. As such, it is necessary to use numerical methods to model these real world problems. In numerical methods, an error term is always present as numerical solutions are only an approximation of the exact solution. It is of interest to test the accuracy of these numerical models.

One popular method from numerical analysis utilized in solving seismic imaging problems is the finite difference method. In this paper, we will employ finite difference methods to model 1D and 2D solutions to the elastic wave equation

$$\rho(\mathbf{x})\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(K(\mathbf{x})\frac{\partial u}{\partial x}\right) \tag{1}$$

where $\rho(\mathbf{x}) = c(\mathbf{x})^{\alpha-2}$ is the density and $K(\mathbf{x}) = c(\mathbf{x})^{\alpha}$ is the bulk modulus for velocity $c(\mathbf{x})$ and parameter α. In order to model these solutions for comparison to the exact solution, we begin by finding the reflection and transmission coefficients for exact solutions to the Eqn. 1 in 1D, 2D, and 3D. We focus on specific traits for the velocity field $c(\mathbf{x})$.

In next section, we describe the specific traits of velocity fields for which we found exact solutions of Eqn. (1) in 1D, 2D, and 3D. In the following sections, we find general equations for the reflection coefficients and transmission coefficients given specific velocity fields in 1D, 2D, and 3D. In the section on the numerical results, we solve for the numerical solutions in 1D and 2D using finite difference methods. In 2D, we consider the case when the wave hits the velocity field a normal incidence. Finally, we discuss future research and conclude.

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REFLECTION PROBLEM

With regards to the velocity field, we focus on a velocity field with two specific traits. First, we are interested in velocities varying only in the z-direction. We restrict the velocity to one-dimension, the z-direction, as that is the traditional downward direction in geophysics, in order to limit the complexity of the problem. As the dimension of the model increases, the more complicated the problem becomes; limiting to the one-dimension makes the problem nicer for computational purposes.

The second trait of interest is when a velocity field has a transition zone. As discussed previously, a velocity $c(z)$ has a transition zone if a portion of $c(z)$ is non-constant. For instance, in (Hardeman and Lamoureux, 2016), we considered a velocity field

$$c(z) = \begin{cases} c_1 & z < \eta; \\ f(z) & \eta \leq z \leq \eta + D; \\ c_2 & \eta + D < z; \end{cases} \tag{2}$$

where c_1, c_2, η are constants, $f(z)$ is an arbitrary continuous function of z, and D is the length of the transition zone. The velocity varies for some function $f(z)$ for $\eta \leq z \leq \eta + D$. As such, we denote this region as our transition zone. For $z < \eta$, we denote this as the top region as it is above the transition zone; for $\eta + D < z$, we have the bottom region because it is below the transition zone. As an illustration, we will consider a velocity jump in each dimension.

With this in mind, we consider what occurs in a general reflection problem. We model this problem using regional solutions. In the top region, there is a plane wave and reflection from the start of the transition region which we write as

$$u_{\text{top}} = u_{\text{inc}} + u_{\text{ref}}. \tag{3}$$

The incident wave $u_{\text{inc}} = u(\mathbf{x}, z, t)$ for some solution u of the elastic wave equation

$$\rho(z)u_{tt} = \nabla(K(z) \cdot \nabla u) \tag{4}$$

where $\rho(z) = c^{\alpha-2}(z)$ and $K(z) = c^{\alpha}(z)$ for some parameter α. Note that $\mathbf{x}$ will be dependent on the dimension we are discussing. For instance, if we are focusing on the 1D case, then $u(\mathbf{x}, z, t) = u(z, t)$. In 2D, $u(\mathbf{x}, z, t) = u(x, z, t)$, and etc. We should note that u_{inc} is positive in z-direction as that is conventionally chosen as the downward direction in geophysical problems. The reflection

$$u_{\text{ref}} = Ru(\mathbf{x}, -z, t) \tag{5}$$

where R is the reflection coefficient. As opposed to u_{inc}, u_{ref} will be negative in the z-direction. Hence, we can consider

$$u_{\text{top}}(\mathbf{x}, z, t) = u(\mathbf{x}, z, t) + Ru(\mathbf{x}, -z, t). \tag{6}$$

We designate what occurs in the transition region as

$$u_{\text{trans}}(\mathbf{x}, z, t) = u(\mathbf{x}, z, t). \tag{7}$$

We employ Separation of Variables to solve the elastic wave equation which separates the PDE into ODEs for each variable. Typically, the solutions to the z-ODE would be of the form $Z(z) = AZ_1(z) + BZ_2(z)$ for arbitrary A, B and functions $Z_1(z)$ and $Z_2(z)$.

In the bottom region, we have the portion of the wave which is transmitted through the transition zone

$$u_{\text{bottom}}(\mathbf{x}, z, t) = Tu(\mathbf{x}, z, t) \tag{8}$$

where T is the transmission coefficient.

The solution u in each of these regional solutions is the portion of the exact solution u of the elastic wave equation corresponding to that particular region.

Given that we are considering piecewise functions, we must enforce some continuity conditions. In particular, we wish to have displacement continuity:

$$u_{\text{top}} = u_{\text{trans}} \quad \text{at } z = \eta \tag{9}$$

$$u_{\text{trans}} = u_{\text{bottom}} \quad \text{at } z = \eta + D \tag{10}$$

and continuity of force:

$$K_{\text{top}}(u_{\text{top}})_z = K_{\text{trans}}(u_{\text{trans}})_z \quad \text{at } z = \eta \tag{11}$$

$$K_{\text{trans}}(u_{\text{trans}})_z = K_{\text{bottom}}(u_{\text{bottom}})_z \quad \text{at } z = \eta + D. \tag{12}$$

After applying these continuity conditions to the regional solutions, we will get a system of equations which we use to solve for the reflection coefficient R.

ONE DIMENSIONAL REFLECTION COEFFICIENTS

In one dimension, we will denote the solutions to the 1D elastic wave equation

$$\rho(z)u_{tt} = (K(z)u_z)_z \tag{13}$$

as $u(z, t)$. For the reflection coefficients in the 1D case, we will extend the work in (Lamoureux et al., 2012) and (Lamoureux et al., 2013) to general solutions.

1D Velocity Jump

In (Lamoureux et al., 2013), the authors found the reflection coefficients when there is a velocity jump

$$c(z) = \begin{cases} c_1 & z < \eta; \\ c_2 & z > \eta \end{cases} \tag{14}$$

where $\eta = 0$. We consider the case for any η. Solving the 1D elastic wave equation with this velocity jump gives the solution

$$u(z, t) = \begin{cases} e^{i\omega(\pm z/c_1 - t)} & z < \eta; \\ e^{i\omega(\pm z/c_2 - t)} & z > \eta. \end{cases} \tag{15}$$

In this case, there is not a transition zone. As such, we define the regional solutions to be

$$u_{\text{top}}(z,t) = e^{i\omega(z/c_1 - t)} + Re^{i\omega(-z/c_1 - t)}; \tag{16}$$

$$u_{\text{bottom}}(z,t) = Te^{i\omega(z/c_2 - t)}. \tag{17}$$

The continuity conditions for this problem are

$$u_{\text{top}} = u_{\text{bottom}} \quad \text{at } z = \eta; \tag{18}$$

$$K_{\text{top}}(u_{\text{top}})_z = K_{\text{bottom}}(u_{\text{bottom}}) \quad \text{at } z = \eta; \tag{19}$$

thus, the system of equations reduces to

$$e^{i\omega\eta/c_1} + Re^{-i\omega\eta/c_1} = Te^{i\omega\eta/c_2} \tag{20}$$

$$c_1^{\alpha-1}\left(e^{i\omega\eta/c_1} - Re^{-i\omega\eta/c_1}\right) = c_2^{\alpha-1}Te^{i\omega\eta/c_2}. \tag{21}$$

Using these equations, we solve to find

$$R(\alpha) = -e^{i2\eta\omega/c_1}\frac{(c_1c_2^{\alpha} - c_1^{\alpha}c_2)}{(c_1c_2^{\alpha} + c_1^{\alpha}c_2)} \text{ and} \tag{22}$$

$$T(\alpha) = \frac{2c_1^{\alpha}c_2e^{i\eta\omega/c_1}e^{-i\eta\omega/c_2}}{c_1c_2^{\alpha} + c_1^{\alpha}c_2}. \tag{23}$$

We will compare this to the results we get in higher dimensions.

1D Velocity Piecewise Ramp

In (Lamoureux et al., 2013), the authors also found the reflection coefficient for the 1D elastic wave equation when a velocity ramp is present. We will consider for general 1D velocity ramps:

$$c(z) = \begin{cases} c_1 & z < \eta; \\ c_{\text{trans}}(z) & \eta \le z \le \eta + D; \\ c_2 & \eta + D < z; \end{cases} \tag{24}$$

where D is the length of the transition zone. In the 1D case, there is only the normal incidence case. As such, we find that solutions to the 1D elastic wave equation are of the form:

$$u(x,t) = \begin{cases} e^{i\omega(\pm z/c_1 - t)} & z < \eta; \\ Z(z)e^{-i\omega t} & \eta \le z \le \eta + D; \\ e^{i\omega(\pm z/c_2 - t)} & \eta + D < z; \end{cases} \tag{25}$$

where $Z(z) = AZ_1(z) + BZ_2(z)$ for arbitrary constants A, B. Given the presence of a transition zone, the regional solutions are

$$u_{\text{top}}(z,t) = e^{i\omega(z/c_1 - t)} + Re^{i\omega(-z/c_1 - t)}; \tag{26}$$

$$u_{\text{trans}}(z,t) = AZ_1(z)e^{-i\omega t} + BZ_2(z)e^{-i\omega t}; \tag{27}$$

$$u_{\text{bottom}}(z,t) = Te^{i\omega(z/c_2 - t)}; \tag{28}$$

which gives the following system of equations when we apply the continuity conditions (9) and (11)

$$e^{i\omega(\eta/c_1-t)} + Re^{i\omega(-\eta/c_1-t)} = (AZ_1(\eta) + BZ_2(\eta))e^{-i\omega t}, \tag{29}$$

$$(AZ_1(\eta+D) + BZ_2(\eta+D))e^{-i\omega t} = Te^{i\omega((\eta+D)/c_2-t)}, \tag{30}$$

$$\frac{i\omega\eta}{c_1}e^{i\omega(\eta/c_1-t)} - \frac{i\omega\eta}{c_1}Re^{i\omega(-\eta/c_1-t)} = (AZ_1'(\eta) + BZ_2'(\eta))e^{-i\omega t}, \text{and} \tag{31}$$

$$(AZ_1'(\eta+D) + BZ_2'(\eta+D))e^{-i\omega t} = \frac{i\omega(\eta+D)}{c_2}Te^{i\omega((\eta+D)/c_2-t)}. \tag{32}$$

We can reduce this system of equations to give

$$e^{i\omega\eta/c_1} + Re^{-i\omega\eta/c_1} = AZ_1(\eta) + BZ_2(\eta), \tag{33}$$

$$AZ_1(\eta+D) + BZ_2(\eta+D) = Te^{i\omega(\eta+D)/c_2}, \tag{34}$$

$$\frac{i\omega}{c_1}e^{i\omega/c_1} - \frac{i\omega}{c_1}Re^{-i\omega\eta/c_1} = AZ_1'(\eta) + BZ_2'(\eta), \text{ and} \tag{35}$$

$$AZ_1'(\eta+D) + BZ_2'(\eta+D) = \frac{i\omega}{c_2}Te^{i\omega(\eta+D)/c_2}. \tag{36}$$

We can make this into the matrix problem:

$$\begin{bmatrix} Z_1(\eta) & Z_2(\eta) & -e^{-i\omega\eta/c_1} & 0 \\ Z_1(\eta+D) & Z_2(\eta+D) & 0 & -e^{i\omega(\eta+D)/c_2} \\ Z_1'(\eta) & Z_2'(\eta) & \frac{i\omega}{c_1}e^{-i\omega\eta/c_1} & 0 \\ Z_1'(\eta+D) & Z_2'(\eta+D) & 0 & -\frac{i\omega}{c_2}e^{i\omega(\eta+D)/c_2} \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} e^{i\omega\eta/c_1} \\ 0 \\ \frac{i\omega}{c_1}e^{i\omega\eta/c_1} \\ 0 \end{bmatrix}. \tag{37}$$

Solving this equation, we get

$$\begin{aligned} R(\omega) = \frac{e^{i2\eta\omega/c_1}}{N_1}&\omega^2(Z_1(D+\eta)Z_2(\eta) - Z_2(D+\eta)Z_1(\eta)) \\ &- c_1c_2(Z_1'(D+\eta)Z_2'(\eta) - Z_2'(D+\eta)Z_1'(\eta)) \\ &+ c_1\omega i(Z_1(D+\eta)Z_2'(\eta) - Z_2(D+\eta)Z_1'(\eta)) \\ &+ c_2\omega i(Z_1'(D+\eta)Z_2(\eta) - Z_2'(D+\eta)Z_1(\eta))), \text{and} \end{aligned} \tag{38}$$

$$\begin{aligned} T(\omega) = &- \frac{i2c_2\omega e^{i\omega(D+\eta)/c_2}e^{i\omega\eta/c_1}}{N1}\times \\ &(Z_1(D+\eta)Z_2'(D+\eta) - Z_2(D+\eta)Z_1'(D+\eta)). \end{aligned} \tag{39}$$

where

$$\begin{aligned} N_1 = &\omega^2(Z_1(D+\eta)Z_2(\eta) - Z_2(D+\eta)Z_1(\eta)) \\ &+ c_1c_2(Z_1'(D+\eta)Z_2'(\eta) - Z_2'(D+\eta)Z_1'(\eta)) \\ &- c_1\omega i(Z_1(D+\eta)Z_2'(\eta) - Z_2(D+\eta)Z_1'(\eta)) \\ &+ c_2\omega i(Z_1'(D+\eta)Z_2(\eta) - Z_2'(D+\eta)Z_1(\eta)). \end{aligned} \tag{40}$$

TWO DIMENSIONAL REFLECTION COEFFICIENTS

For the two dimensional case, we will denote the solutions to the 2D elastic wave equation

$$\rho(z)u_{tt} = \nabla(K(z)\nabla u) = K(z)u_{zz} + K'(z)u_z \tag{41}$$

where $\nabla = (\partial_x, \partial_z)$ as $u(x, z, t)$. In the following sections, we will consider a velocity jump first and the general case of a piecewise velocity with a transition zone which can be applied to solutions of the 2D elastic wave equation with piecewise velocities.

2D Velocity Jump

First, we consider a 2D velocity jump

$$c(z) = \begin{cases} c_1 & z < \eta; \\ c_2 & z > \eta; \end{cases} \tag{42}$$

where c_1 and c_2 are constant. We discuss this velocity here in order to compare the reflection provided when a ramp is present versus when one is not present.

For this case, a plane wave moves downward across a velocity jump. Thus, we represent the wave in the top half by

$$u_{\text{top}}(x, z, t) = e^{i\omega(z/c_1 - t)} + Re^{i\omega(-z/c_1 - t)} \tag{43}$$

and the wave which is transmitted across the velocity jump as

$$u_{\text{bottom}}(x, z, t) = Te^{i\omega(z/c_2 - t)}, \tag{44}$$

where R is the reflection coefficient and T is the transmission coefficient.

Again, we want to enforce displacement continuity across the jump as well as continuity of force. As such, we consider the continuity conditions

$$u_{\text{top}} = u_{\text{bottom}} \quad \text{at } z = \eta; \tag{45}$$

$$K_{\text{top}}\nabla(u_{\text{top}}) = K_{\text{bottom}}\nabla(u_{\text{bottom}}) \quad \text{at } z = \eta; \tag{46}$$

resulting in

$$e^{i\omega(\eta/c_1 - t)} + Re^{i\omega(-\eta/c_1 - t)} = Te^{i\omega(\eta/c_2 - t)}; \tag{47}$$

$$K_{\text{top}}\left(\frac{i\omega\eta}{c_1}e^{i\omega(\eta/c_1 - t)} - \frac{i\omega\eta}{c_1}Re^{i\omega(-\eta/c_1 - t)}\right) = K_{\text{bottom}}\left(T\frac{i\omega\eta}{c_2}e^{i\omega(\eta/c_2 - t)}\right); \tag{48}$$

which implies

$$e^{i\omega\eta/c_1} + Re^{-i\omega\eta/c_1} = Te^{i\omega\eta/c_2} \tag{49}$$

$$K_{\text{top}}\left(\frac{e^{i\omega\eta/c_1}}{c_1} - R\frac{e^{-i\omega\eta/c_1}}{c_1}\right) = K_{\text{bottom}}\left(\frac{e^{i\omega\eta/c_2}}{c_2}T\right) \tag{50}$$

Recall that $K(z) = c^{\alpha}(z)$. Then,

$$e^{i\omega\eta/c_1} + Re^{-i\omega\eta/c_1} = Te^{i\omega\eta/c_2} \tag{51}$$

$$c_1^{\alpha-1}\left(e^{i\omega\eta/c_1} - Re^{-i\omega\eta/c_1}\right) = c_2^{\alpha-1}\left(Te^{i\omega\eta/c_2}\right). \tag{52}$$

This becomes the matrix problem

$$\begin{bmatrix} -e^{-i\omega\eta/c_1} & e^{i\omega(\eta/c_2)} \\ c_1^{\alpha-1}e^{-i\omega\eta/c_1} & c_2^{\alpha-1}e^{i\omega\eta/c_2} \end{bmatrix}\begin{bmatrix} R \\ T \end{bmatrix} = \begin{bmatrix} e^{i\omega\eta/c_1} \\ c_1^{\alpha-1}e^{i\omega\eta/c_1} \end{bmatrix} \tag{53}$$

which we can solve to get

$$R(\alpha) = -e^{i2\eta\omega/c_1}\frac{(c_1c_2^{\alpha} - c_1^{\alpha}c_2)}{(c_1c_2^{\alpha} + c_1^{\alpha}c_2)} \text{ and} \tag{54}$$

$$T(\alpha) = \frac{2c_1^{\alpha}c_2e^{i\eta\omega/c_1}e^{-i\eta\omega/c_2}}{c_1c_2^{\alpha} + c_1^{\alpha}c_2}. \tag{55}$$

Given that we are considering the plane wave case in 2D where the wave is orthogonal to the velocity jump, it follows that the exact solutions for reflection and transmission coefficients would mimic the 1D case. In fact, the 2D reflection coefficient solution (Eqn. 54) is identical to the 1D reflection coefficient solution (Eqn. 22) for this case.

2D Piecewise Velocity Ramp

We now consider what occurs for general 2D piecewise velocity ramps

$$c(z) = \begin{cases} c_1 & z < \eta \\ c_{\text{trans}}(z) & \eta \le z \le \eta + D \\ c_2 & \eta + D < z. \end{cases} \tag{56}$$

Unlike in the 1D case, we now have to consider what occurs in two directions: the x-direction and the z-direction. The plane wave can hit the ramp at normal incidence or non-normal incidence.

In the normal incidence case, the plane wave is orthogonal to the transition zone of the velocity field. This means that the incident angle $\theta = 0°$. As such, for the 2D problem of normal incidence, the plane wave is constant in the x-direction. Hence, the solution to Eqn. (41) is

$$u(x,z,t) = \begin{cases} e^{i\omega(\pm z/c_1 - t)} & z < \eta,\ x \in \mathbb{R}; \\ Z(z)e^{-i\omega t} & \eta \le z \le \eta + D,\ x \in \mathbb{R}; \\ e^{i\omega(\pm z/c_2 - t)} & \eta + D < z,\ x \in \mathbb{R}; \end{cases} \tag{57}$$

where $Z(z) = AZ_1(z) + BZ_1(z)$ for arbitrary A, B. Then, the regional solutions for this case are

$$u_{\text{top}}(x,z,t) = e^{i\omega(z/c_1 - t)} + Re^{i\omega(-z/c_1 - t)}; \tag{58}$$

$$u_{\text{trans}}(x,z,t) = AZ_1(z)e^{-i\omega t} + BZ_2(z)e^{-i\omega t}; \tag{59}$$

$$u_{\text{bottom}}(x,z,t) = Te^{i\omega(z/c_2 - t)}. \tag{60}$$

Applying the 2D continuity conditions, we get the following system of equations

$$e^{i\omega\eta/c_1} + Re^{-i\omega\eta/c_1} = AZ_1(\eta) + BZ_2(\eta); \tag{61}$$

$$AZ_1(\eta+D) + BZ_2(\eta+D) = Te^{i\omega(\eta+D)/c_2}; \tag{62}$$

$$c_1^\alpha\left(\frac{i\omega}{c_1}e^{i\omega\eta/c_1} - \frac{i\omega}{c_1}Re^{-i\omega\eta/c_1}\right) = c_1^\alpha\left(AZ_1'(\eta) + BZ_2'(\eta)\right); \tag{63}$$

$$c_2^\alpha\left(AZ_1'(\eta+D) + BZ_2'(\eta+D)\right) = c_2^\alpha T\frac{i\omega}{c_2}e^{i\omega(\eta+D)/c_2}; \tag{64}$$

which gives us the matrix problem

$$\begin{bmatrix} Z_1(\eta) & Z_2(\eta) & -e^{-i\omega\eta/c_1} & 0 \\ Z_1(\eta+D) & Z_2(\eta+D) & 0 & -e^{i\omega(\eta+D)/c_2} \\ Z_1'(\eta) & Z_2'(\eta) & \frac{i\omega}{c_1}e^{i\omega\eta/c_1} & 0 \\ Z_1'(\eta+D) & Z_2'(\eta+D) & 0 & -\frac{i\omega}{c_2}e^{i\omega(\eta+D)/c_2} \end{bmatrix}\begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} e^{i\omega\eta/c_1} \\ 0 \\ \frac{i\omega}{c_1}e^{i\omega\eta/c_1} \\ 0 \end{bmatrix}. \tag{65}$$

Solving the matrix equation, we get

$$\begin{aligned} R(\omega) = &-\frac{ie^{i2\eta\omega/c_1}}{N_2}[c_1c_2(Z_1'(D+\eta)Z_2'(\eta) - Z_2'(D+\eta)Z_1'(\eta)) \\ &- \omega^2 e^{i2\omega(D+\eta)/c_2}(Z_1'(D+\eta)Z_2(\eta) - Z_2(D+\eta)Z_1(\eta)) \\ &- ic_2\omega(Z_1'(D+\eta)Z_2(\eta) - Z_2'(D+\eta)Z_1(\eta)) \\ &- ic_1\omega e^{i2\omega(D+\eta)/c_2}(Z_1(D+\eta)Z_2'(\eta) - Z_2(D+\eta)Z_1'(\eta))], \text{ and} \end{aligned} \tag{66}$$

$$\begin{aligned} T(\omega) = &\frac{c_2\omega e^{i\omega[(D+\eta)/c_2+\eta/c_1]}}{N_2}(1+e^{i2\eta\omega/c_1}) \\ &(Z_1(D+\eta)Z_2'(D+\eta) - Z_2(D+\eta)Z_1'(D+\eta)) \end{aligned} \tag{67}$$

where

$$\begin{aligned} N_2 = &ic_1c_2(Z_1'(D+\eta)Z_2'(\eta) - Z_2'(D+\eta)Z_1'(\eta)) \\ &+ c_1\omega e^{i2\omega(D+\eta)/c_2}(Z_1(D+\eta)Z_2'(\eta) - Z_2(D+\eta)Z_1'(\eta)) \\ &+ i\omega^2 e^{i2\omega(D+\eta)/c_2}e^{i2\eta\omega/c_1}(Z_1(D+\eta)Z_2'(\eta) - Z_2(D+\eta)Z_1(\eta)) \\ &- c_2\omega e^{i2\eta\omega/c_1}(Z_1'(D+\eta)Z_2(\eta) - Z_2'(D+\eta)Z_1(\eta)). \end{aligned} \tag{68}$$

For the non-normal incidence case, the plane wave hits the transition zone at an angle $\theta_1 \geq 0$. In normal incidence case, the wave was constant in the x-direction; however, that is not true for this case. As such, the solution to Eqn. (41) is

$$u(x,z,t) = \begin{cases} e^{i(k_x x \pm k_z z - \omega t} & z < \eta; \\ Z(z)e^{i(k_x x - \omega t)} & \eta \leq z \leq \eta + D; \\ e^{i(k_x x \pm k_z' z - \omega t)} & \eta + D < z; \end{cases} \tag{69}$$

where $Z(z) = AZ_1(z) + BZ_2(z)$ with arbitrary A, B. The regional solutions are given by

$$u_{\text{top}}(x,z,t) = e^{i(k_x x + k_z z - \omega t)} + Re^{i(k_x x - k_z z - \omega t)}; \tag{70}$$

$$u_{\text{trans}}(x,z,t) = (AZ_1(z) + BZ_2(z))e^{i(k_x x - \omega t)}; \tag{71}$$

$$u_{\text{bottom}}(x,z,t) = Te^{i(k_x x + k_z' z - \omega t)}. \tag{72}$$

Applying the 2D continuity conditions and reducing, we get the system of equations

$$e^{ik_z\eta} + Re^{-ik_z\eta} = AZ_1(\eta) + BZ_2(\eta); \tag{73}$$
$$AZ_1(\eta + D) + BZ_2(\eta + D) = Te^{ik_z'(\eta+D)}; \tag{74}$$
$$ik_z e^{ik_z\eta} - iRk_z e^{-ik_z\eta} = AZ_1'(\eta) + BZ'(\eta); \tag{75}$$
$$AZ_1'(\eta + D) + BZ_2'(\eta + D) = ik_z' Te^{ik_z'(\eta+D)}; \tag{76}$$

which gives the matrix problem:

$$\begin{bmatrix} Z_1(\eta) & Z_2(\eta) & -e^{-ik_z\eta} & 0 \\ Z_1(\eta+D) & Z_2(\eta+D) & 0 & -e^{ik_z'(\eta+D)} \\ Z_1'(\eta) & Z_2'(\eta) & ik_z e^{-ik_z\eta} & 0 \\ Z_1'(\eta+D) & Z_2'(\eta+D) & 0 & -ik_z' e^{ik_z'(\eta+D)} \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} e^{ik_z\eta} \\ 0 \\ ik_z e^{ik_z\eta} \\ 0 \end{bmatrix}. \tag{77}$$

Solving this problem, we get

$$\begin{aligned} R(\omega) = \frac{e^{\eta k_z 2i}}{N_3} [&Z_2'(D+\eta)Z_1'(\eta) - Z_1'(D+\eta)Z_2'(\eta) \\ &+ ik_z(Z_1'(D+\eta)Z_2(\eta)i - Z_2'(D+\eta)Z_1(\eta)) \\ &+ ik_z'(Z_1(D+\eta)Z_2'(\eta) - Z_2(D+\eta)Z_1'(\eta)) \\ &+ k_z k_z'(Z_1(D+\eta)Z_2(\eta) - Z_2(D+\eta)Z_1(\eta))], \end{aligned} \tag{78}$$

$$T(\omega) = \frac{-2ik_z e^{-ik_z'(D+\eta)} e^{i\eta k_z}}{N_3} (Z_1(D+\eta)Z_2'(D+\eta) - Z_2(D+\eta)Z_1'(D+\eta)) \tag{79}$$

where

$$\begin{aligned} N_3 = {} & Z_1'(D+\eta)Z_2'(\eta) - Z_2'(D+\eta)Z_1'(\eta) \\ &+ ik_z(Z_1'(D+\eta)Z_2(\eta) - Z_2'(D+\eta)Z_1(\eta)) \\ &- ik_z'(Z_1(D+\eta)Z_2'(\eta)i - Z_2(D+\eta)Z_1'(\eta)) \\ &+ k_z k_z'(Z_1(D+\eta)Z_2(\eta) - Z_2(D+\eta)Z_1(\eta)). \end{aligned} \tag{80}$$

THREE DIMENSIONAL REFLECTION COEFFICIENTS

For the three dimensional case, we will denote the solutions to the 3D elastic wave equation

$$\rho(z)u_{tt} = \nabla(K(z)\nabla u) \tag{81}$$

where $\nabla = (\partial_x, \partial_y, \partial_z)$ and $u = u(x, y, z, t)$. In the next few sections, we focus on finding reflection coefficients given a 3D velocity jump and a general 3D velocity ramp.

3D Velocity Jump

First, we consider a 3D velocity jump

$$c(z) = \begin{cases} c_1 & z < \eta; \\ c_2 & z > \eta; \end{cases} \tag{82}$$

where c_1 and c_2 are constant. For this case, a plane wave moves downward across a velocity jump. Thus, we represent the wave in the top half by

$$u_{\text{top}}(x, y, z, t) = e^{i\omega(z/c_1 - t)} + Re^{i\omega(-z/c_1 - t)} \tag{83}$$

and the wave which is transmitted across the velocity jump as

$$u_{\text{bottom}}(x, y, z, t) = Te^{i\omega(z/c_2 - t)}, \tag{84}$$

where R is the reflection coefficient and T is the transmission coefficient.

These are the same regional solutions as the 2D velocity jump. See Eqns. (43) and (44). As such, the reflection and transmission coefficients will be the same for the 3D as in 2D case. See Eqn. (54). It will be interesting to see if the 3D velocity ramp mimics the 2D case as well.

3D Piecewise Velocity Ramp

In this section, we will discuss the reflection coefficient given a general 3D piecewise velocity ramp

$$c(z) = \begin{cases} c_1 & z < \eta; \\ c_{\text{trans}}(z) & \eta \le z \le \eta + D; \\ c_2 & \eta + D < z; \end{cases} \tag{85}$$

where $c_{\text{trans}}(z)$ is the velocity ramp and D is the length of the ramp.

In the normal incidence case, we get the solution

$$u(x, y, z, t) = \begin{cases} e^{i\omega(z/c_1 - t)} & z < \eta,\ x, y \in \mathbb{R}; \\ Z(z)e^{-i\omega t} & \eta \le z \le \eta + D,\ x, y \in \mathbb{R}; \\ e^{i\omega(z/c_2 - t)} & \eta + D < z,\ x, y \in \mathbb{R}; \end{cases} \tag{86}$$

where $Z(z) = AZ_1(z) + BZ_2(z)$ for arbitrary A, B. The regional solutions for this case are

$$u_{\text{top}}(x, y, z, t) = e^{i\omega(z/c_1 - t)} + Re^{-i\omega(z/c_1 - t)}; \tag{87}$$

$$u_{\text{trans}}(x, y, z, t) = (AZ_1(z) + BZ_2(z))e^{-i\omega t}; \text{ and} \tag{88}$$

$$u_{\text{bottom}}(x, y, z, t) = Te^{i\omega(z/c_2 - t)}; \tag{89}$$

where when we apply the 3D continuity conditions, we get

$$e^{i\omega(\eta/c_1 - t)} + Re^{-i\omega(\eta/c_1 - t)} = (AZ_1(\eta) + BZ_2(\eta))e^{-i\omega t}; \tag{90}$$

$$(AZ_1(\eta + D) + BZ_2(\eta + D))e^{-i\omega t} = Te^{i\omega((\eta + D)/c_2 - t)}; \tag{91}$$

$$\frac{i\omega\eta}{c_1}e^{i\omega(\eta/c_1 - t)} - \frac{i\omega\eta}{c_1}Re^{-i\omega(\eta/c2 - t)} = (AZ_1'(\eta) + BZ_2'(\eta))e^{-i\omega t}; \tag{92}$$

$$(AZ_1'(\eta + D) + BZ_2'(\eta + D))e^{-i\omega t} = T\frac{i\omega(\eta + D)}{c_2}e^{i\omega((\eta + D)/c_2 - t)}. \tag{93}$$

We can reduce this to

$$e^{i\omega\eta/c_1} + Re^{-i\omega\eta/c_1} = AZ_1(\eta) + BZ_2(\eta); \tag{94}$$

$$AZ_1(\eta + D) + BZ_2(\eta + D) = Te^{i\omega(\eta+D)/c_2}; \tag{95}$$

$$\frac{i\omega\eta}{c_1}e^{i\omega\eta/c_1} - \frac{i\omega\eta}{c_1}Re^{-i\omega\eta/c_2} = AZ_1'(\eta) + BZ_2'(\eta); \tag{96}$$

$$AZ_1'(\eta + D) + BZ_2'(\eta + D) = T\frac{i\omega(\eta + D)}{c_2}e^{i\omega(\eta+D)/c_2}; \tag{97}$$

which becomes the matrix problem

$$\begin{bmatrix} Z_1(\eta) & Z_2(\eta) & e^{-i\omega\eta/c_1} & 0 \\ Z_1(\eta+D) & Z_2(\eta+D) & 0 & -e^{i\omega(\eta+D)/c_2} \\ Z_1'(\eta) & Z_2'(\eta) & \frac{i\omega}{c_1}e^{-i\omega\eta/c_2} & 0 \\ Z_1'(\eta+D) & Z_2'(\eta+D) & 0 & -\frac{i\omega}{c_2}e^{i\omega(\eta+D)/c_2} \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} e^{i\omega\eta/c_1} \\ 0 \\ \frac{i\omega}{c_1}e^{i\omega\eta/c_1} \\ 0 \end{bmatrix}. \tag{98}$$

This is the same matrix problem as for the 2D case Eqn. (65). As such, the exact solution for the reflection coefficient will be the same.

In the non-normal incidence case, the 3D solutions to the elastic wave equation are of the form:

$$u(x, y, z, t) = \begin{cases} e^{i(k_x x + k_y y \pm k_z z - \omega t)} & z < \eta,\ x, y \in \mathbb{R}; \\ (AZ_1(z) + BZ_2(z))e^{i(k_x x + k_y y - \omega t)} & \eta \le z \le \eta + D,\ x, y \in \mathbb{R}; \\ e^{i(k_x x + k_y y \pm k_z' z - \omega t)} & \eta + D < z,\ x, y \in \mathbb{R}. \end{cases} \tag{99}$$

Therefore, the regional solutions for this case are

$$u_{\text{top}}(x, y, z, t) = e^{i(k_x x + k_y y + k_z z - \omega t)} + Re^{i(k_x x + k_y y - k_z z - \omega t)}, \tag{100}$$

$$u_{\text{trans}}(x, y, z, t) = (AZ_1(z) + BZ_2(z))e^{i(k_x x + k_y y - \omega t)}, \text{ and} \tag{101}$$

$$u_{\text{bottom}}(x, y, z, t) = Te^{i(k_x x + k_y y + k_z' z - \omega t)}. \tag{102}$$

Applying the continuity conditions, we get the system

$$e^{i(k_x x + k_y y - \omega t)}(e^{ik_z\eta} + Re^{-ik_z\eta}) = (AZ_1(\eta) + BZ_2(\eta))e^{i(k_x x + k_y y - \omega t)}; \tag{103}$$

$$(AZ_1(\eta + D) + BZ_2(\eta + D))e^{i(k_x x + k_y y - \omega t)} = Te^{i(k_x x + k_y y + k_z'(\eta+D) - \omega t)}; \tag{104}$$

$$c_1^\alpha e^{i(k_x x + k_y y - \omega t)}(ik_z e^{ik_z\eta} - ik_z Re^{-ik_z\eta}) = c_1^\alpha(AZ_1'(\eta) + BZ_2'(\eta))e^{i(k_x x + k_y y - \omega t)}; \tag{105}$$

$$c_2^\alpha(AZ_1'(\eta + D) + BZ_2'(\eta + D))e^{i(k_x x + k_y y - \omega t)} = c_2^\alpha ik_z' Te^{i(k_x x + k_y y + k_z'(\eta+D) - \omega t)}; \tag{106}$$

which reduces to

$$e^{ik_z\eta} + Re^{-ik_z\eta} = AZ_1(\eta) + BZ_2(\eta); \tag{107}$$

$$AZ_1(\eta + D) + BZ_2(\eta + D) = Te^{ik_z'(\eta+D)}; \tag{108}$$

$$ik_z e^{ik_z\eta} - ik_z Re^{-ik_z\eta} = AZ_1'(\eta) + BZ_2'(\eta); \tag{109}$$

$$AZ_1'(\eta + D) + BZ_2'(\eta + D) = ik_z' Te^{ik_z'(\eta+D)}. \tag{110}$$

We can make this into the matrix problem

$$\begin{bmatrix} Z_1(\eta) & Z_2(\eta) & -e^{-ik_z\eta} & 0 \\ Z_1(\eta+D) & Z_2(\eta+D) & 0 & -e^{ik'_z(\eta+D)} \\ Z'_1(\eta) & Z'_2(\eta) & ik_z e^{-ik_z\eta} & 0 \\ Z'_1(\eta+D) & Z'_2(\eta+D) & 0 & -ik'_z e^{ik'_z(\eta+D)} \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} e^{ik_z\eta} \\ 0 \\ ik_z e^{ik'_z\eta} \\ 0. \end{bmatrix} \tag{111}$$

Recall this is the same matrix as for the 2D case. See Eqn. (77). Therefore, the equation for the reflection and transmission coefficients will be the same.

COMPARISON TO NUMERICAL SOLUTIONS

In the geosciences, it is common to model solutions to the elastic wave equation using numerical methods. In this section, we will compare the exact solutions of the elastic wave equation given a velocity ramp and some parameter α to numerical solutions modeled using finite difference methods. We employ the equations we found in the previous sections for the reflection and transmission coefficients in our finite difference scheme. First, we will consider the 1D case. Then, we will compare the exact and numerical solutions in the 2D normal incidence case given a velocity ramp and some parameter α.

1D Numerical Results

Consider a velocity ramp

$$c(z) = \begin{cases} 1 & z < 1; \\ z & 1 \leq z \leq 2; \\ 2 & 2 < z. \end{cases} \tag{112}$$

For the 1D model, we made a 1001×6001 grid for $z = -1, ..., 5$ and $t = 0, ..., 4$. We set parameter $\alpha = 2$. Using the information from previous sections, we calculated the approximated solution given the reflection coefficient for the 1D velocity ramp (112). Fig. 1 shows the real part of the the exact and approximated solutions. Fig. 2 depicts the imaginary part of the exact and approximated solutions. In both solutions, there is a distortion between $z = 1$ and $z = 2$. This distortion represents the velocity ramp (112).

In Fig. 3, we see the residuals for the real and imaginary parts. The difference between the exact and approximated solutions in both the real and imaginary parts is a little greater than 0.03.

2D Numerical Results

For the 2D case, we extend the linear velocity ramp (112) to

$$c(z) = \begin{cases} 1 & z < 1; \\ z & 1 \leq z \leq 2; \\ 2 & 2 < z. \end{cases} \tag{113}$$

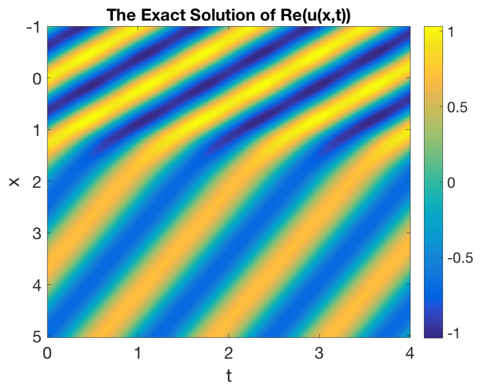

FIG. 1. (Left) The real part of the exact solution in 1D. (Right) The real part of the approximated solution in 1D.

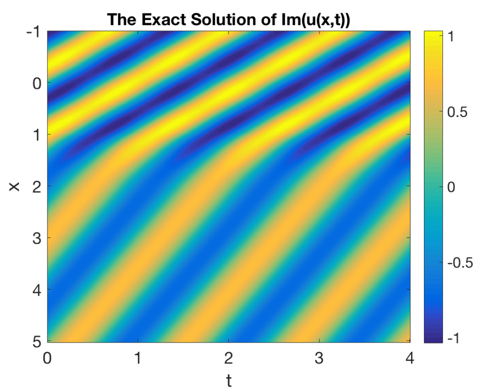

FIG. 2. (Left) The imaginary part of the exact solution in 1D. (Right) The imaginary part of the approximated solution in 1D.

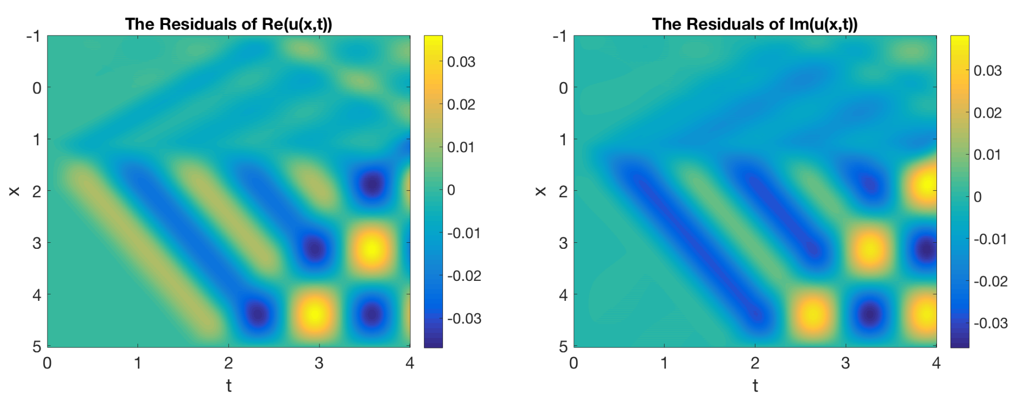

FIG. 3. (Left) The residuals from the real part of the exact and approximated solutions. (Right) The residuals from the imaginary part of the exact and approximated solutions.

Given the 2D velocity (113), we consider what occurs when the wave is normal to the velocity ramp (113) in the 2D case. We created a 3D grid which is $101 \times 101 \times 601$ where $z = -1, ..., 5$, $x = -1, ..., 5$, and $t = 0, ..., 4$. We set the parameter α equal to 2.

In Fig. 4 and 5, we see the real part of the exact and approximated solutions to the 2D elastic wave equation at time $t = 0, 1, 2, 3$, and 4 respectively. Fig. 6 and 7 show the imaginary part of the exact and approximated solutions to the 2D elastic wave equation at $t = 0, 1, 2, 3$, and 4, respectively. In both the real and imaginary parts, we can see the waves move further apart after passing through the transition zone $z = 1$ to $z = 2$ similar to the distortion that occurred in the transition zone in the 1D case.

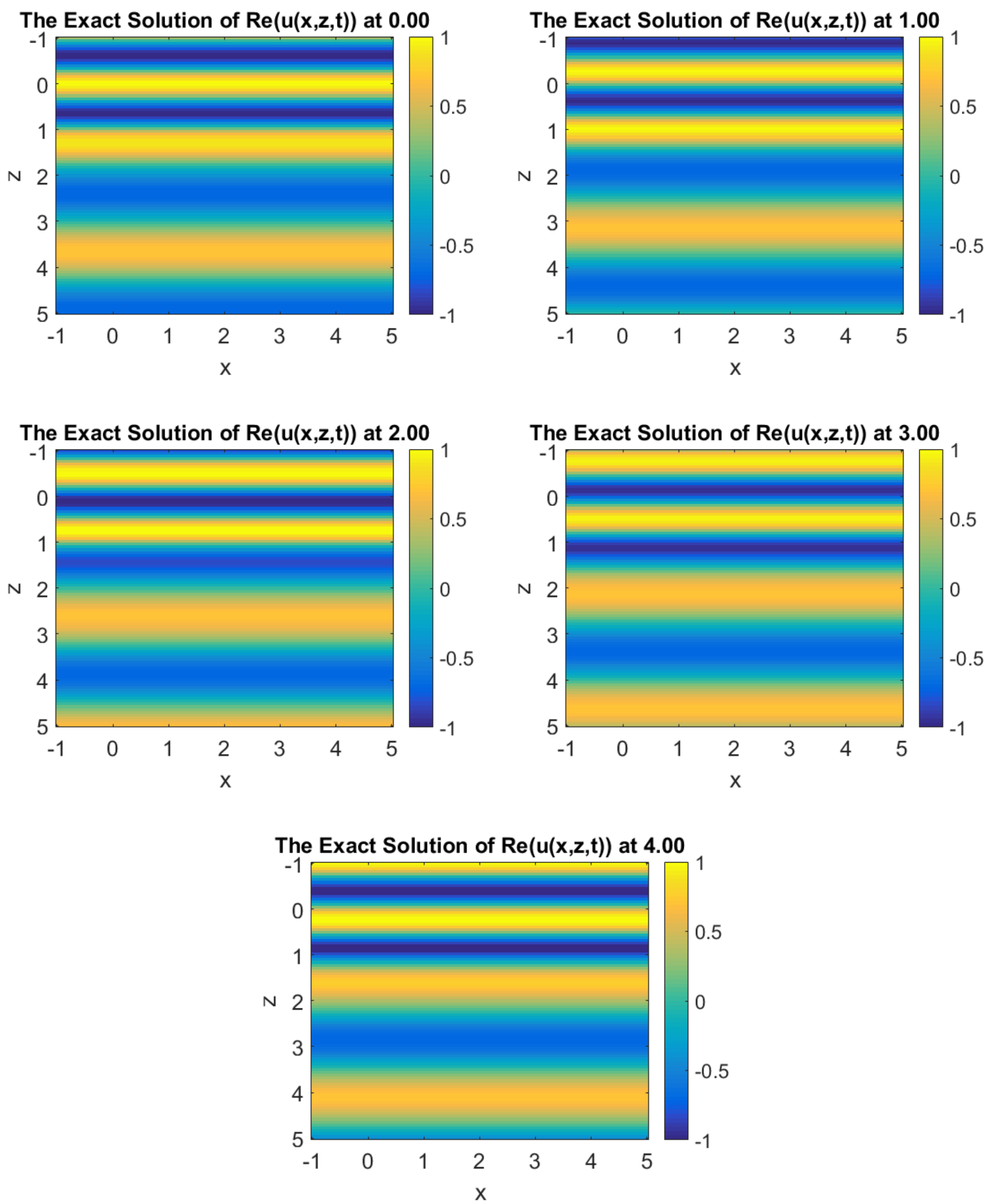

FIG. 4. The real part of the exact solutions at (top left) $t = 0$, (top right) $t = 1$, (middle left) $t = 2$, (middle right) $t = 3$, and (bottom) $t = 4$.

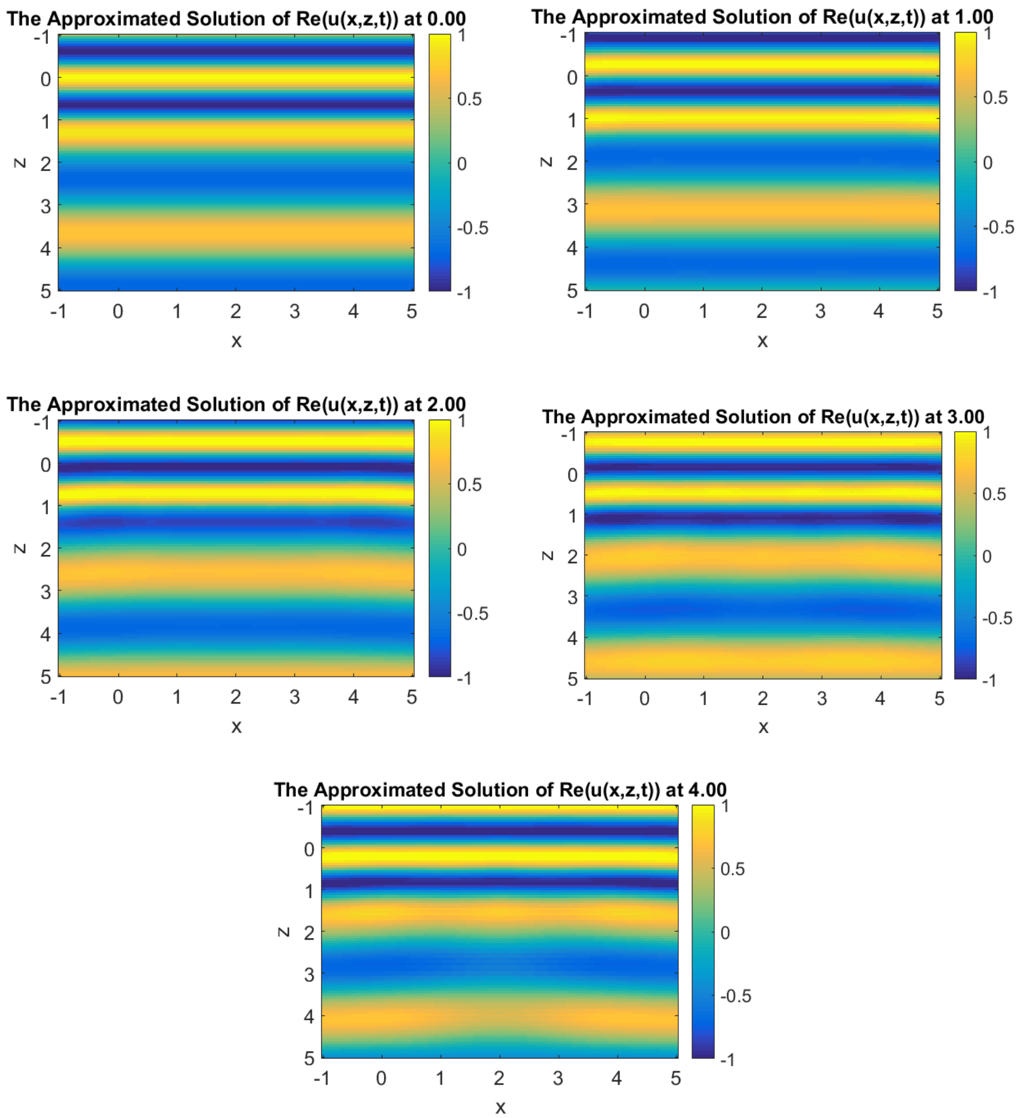

FIG. 5. The real part of the approximated solutions at (top left) $t = 0$, (top right) $t = 1$, (middle left) $t = 2$, (middle right) $t = 3$, and (bottom) $t = 4$.

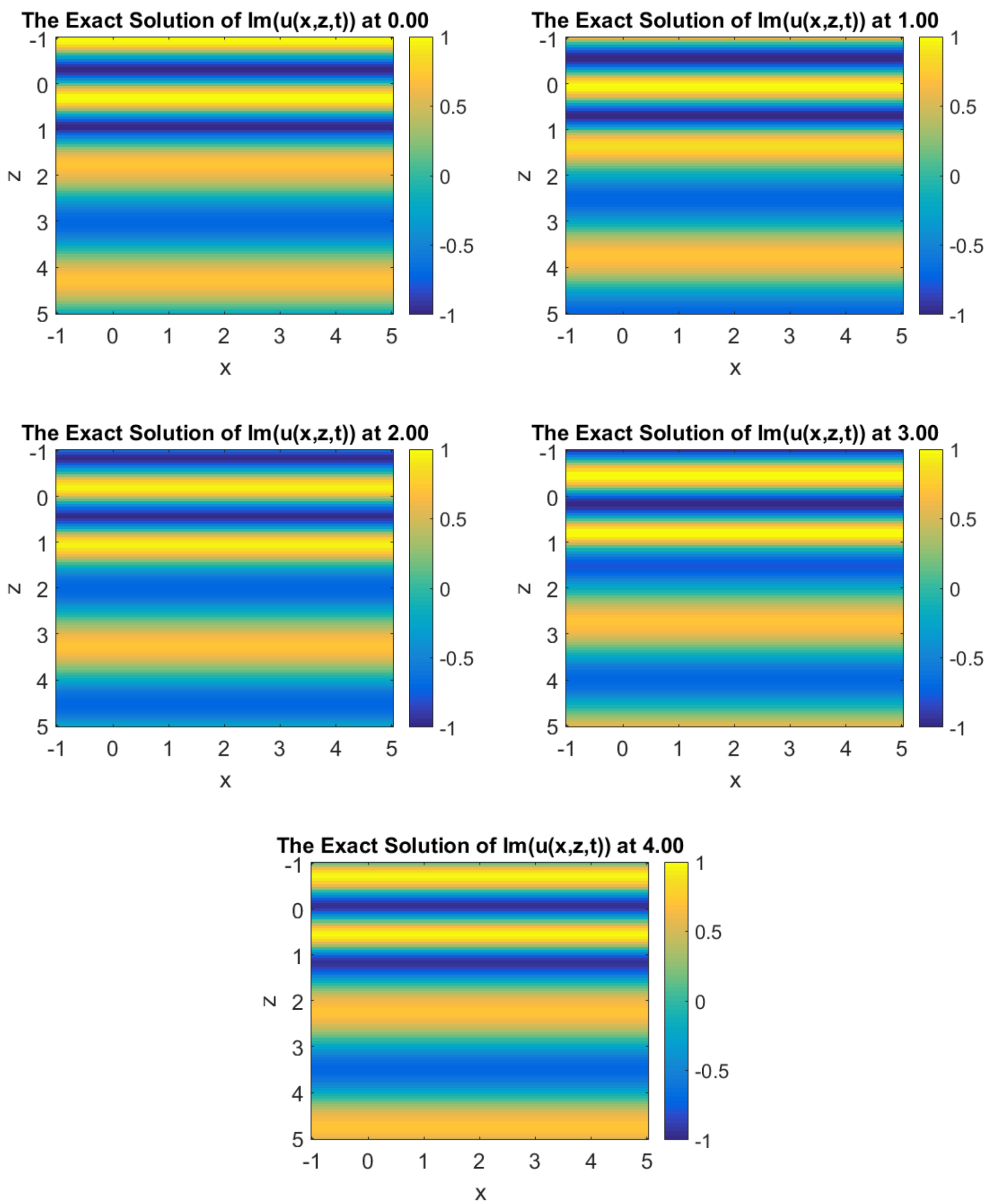

FIG. 6. The imaginary part of the exact solutions at (top left) $t = 0$, (top right) $t = 1$, (middle left) $t = 2$, (middle right) $t = 3$, and (bottom) $t = 4$.

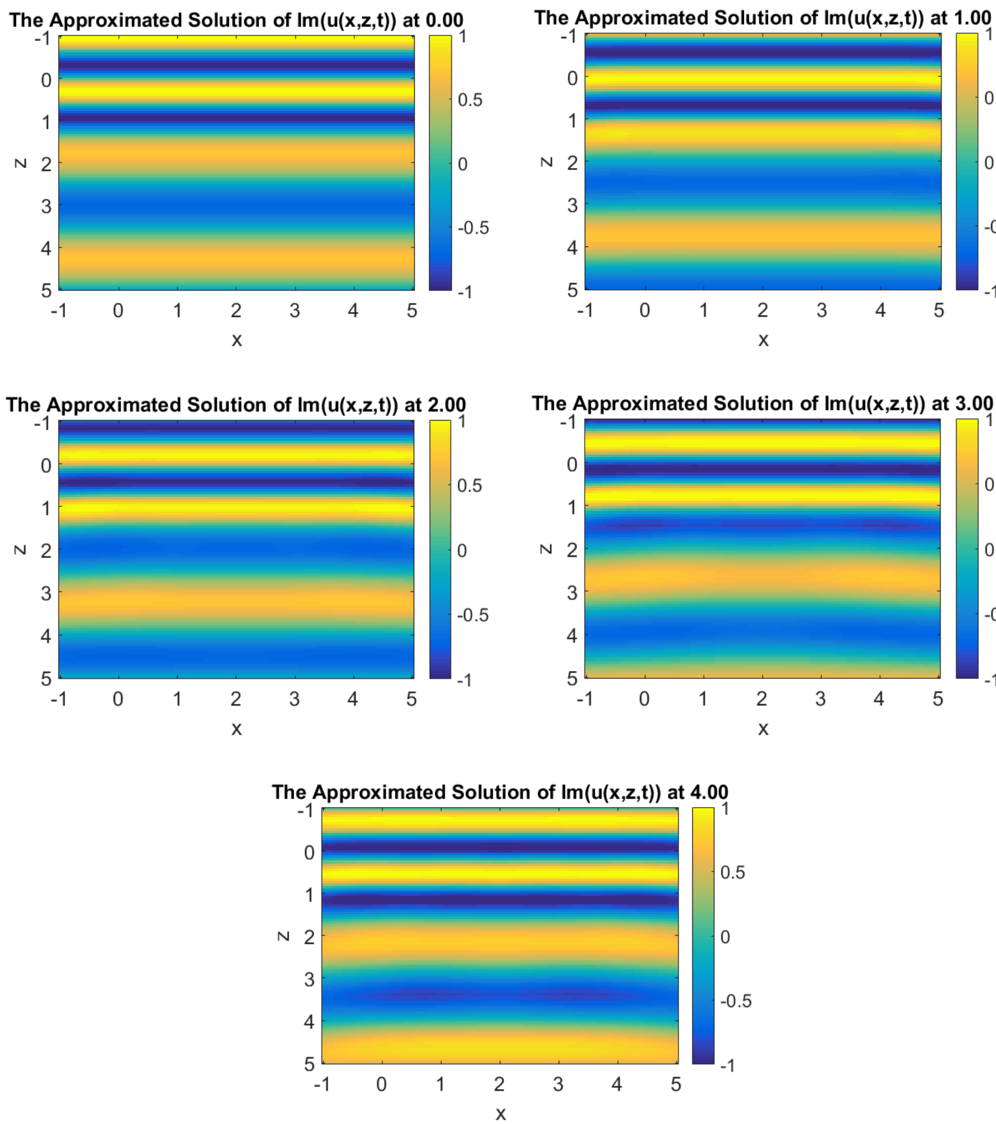

FIG. 7. The imaginary part of the exact and approximated solutions at (top left) $t = 0$, (top right) $t = 1$, (middle left) $t = 2$, (middle right) $t = 3$, and (bottom) $t = 4$.

Fig. 8 and 9 depicts the difference between the real and imaginary parts of the exact and approximated solutions at $t = 0, 1, 2, 3$, and 4. In both cases, we see that the difference between the exact and approximated solutions is less than 0.2. As such, between 1D and 2D, the error increased by a factor of approximately 7.

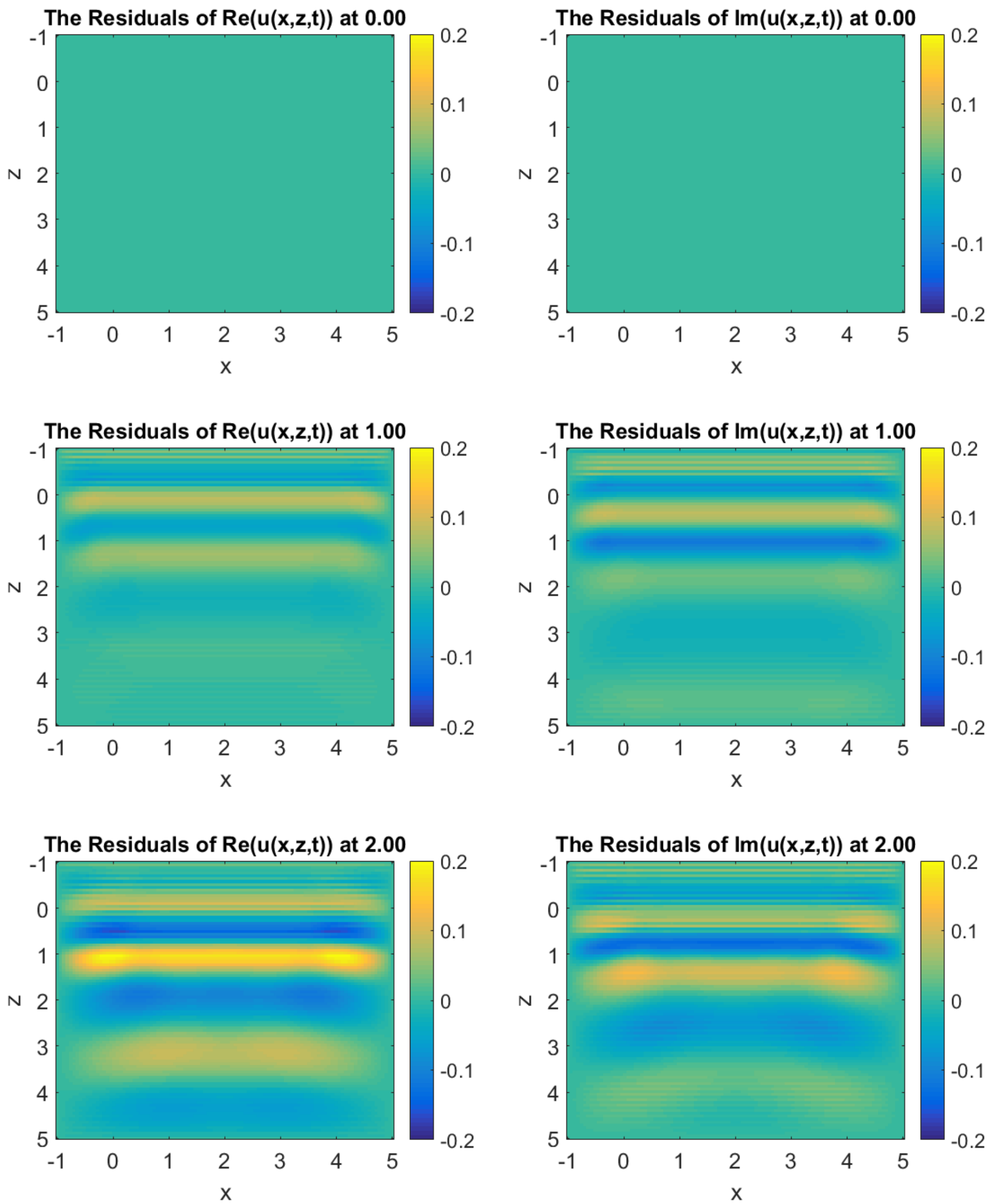

FIG. 8. The residuals from the real part of the exact and approximated solutions at (top left), (middle left), and (bottom left). The residuals from the imaginary part of the exact and approximated solutions (top right), (middle right), and (bottom right).

FUTURE WORK

Our next step will be to consider what occurs in the 2D case when the wave hits the velocity ramp at non-normal incidence. From there, we would extend the results to 3D for both the normal and non-normal incidence cases.

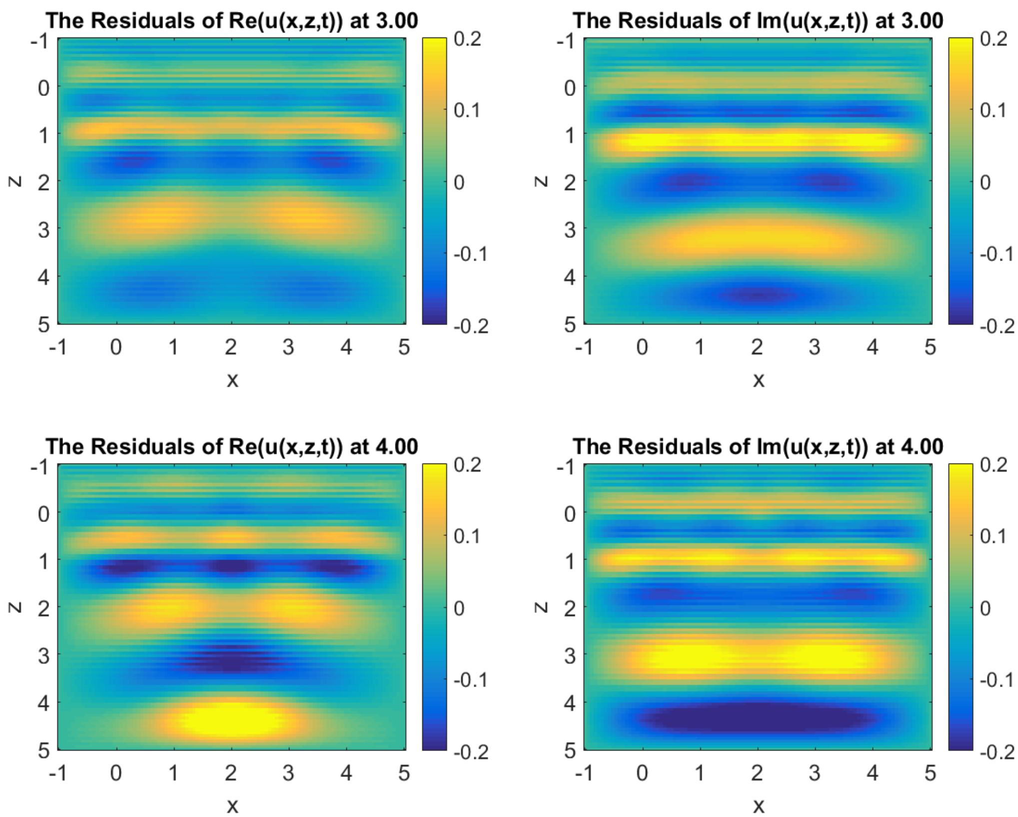

FIG. 9. The residuals from the real part of the exact and approximated solutions (top left), and (bottom left). The residuals from the imaginary part of the exact and approximated solutions (top right), and (bottom right).

CONCLUSIONS

We found general solutions for reflections and transmission coefficients given a velocity jump and a piecewise velocity ramp in 1D, 2D, and 3D. For the velocity jump, we found that the equations were equivalent in all three dimensions. In 2D and 3D, we considered the results for normal incidence and non-normal incidence when a velocity ramp was present. We noted that in the 2D and 3D cases the general equations for reflection and transmission coefficients were equivalent. The main difference would arise from the ODE solutions $Z_1(z)$ and $Z_2(z)$ in the different dimensions. Finally, we employed the general equations for reflection and transmission coefficients as well as the variables A and B in order to compare the results of exact solutions to the elastic wave equation given a linear velocity ramp and some parameter α to numerical solutions modeled using finite difference methods. We also considered the residuals of each case and saw that the residuals increased by a factor of about 7 between the 1D case and the 2D case.

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