Minimum Phase Revisited

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Introduction

- Minimum phase for digital systems
- Fourier synthesis: an example
- Tempered distributions
- Causality and the Hilbert transform
- Minimum phase for analog systems
- Examples
- Conclusions

Minimum phase
for Digital systems

Minimum phase for Digital systems

A minimum phase digital signal…

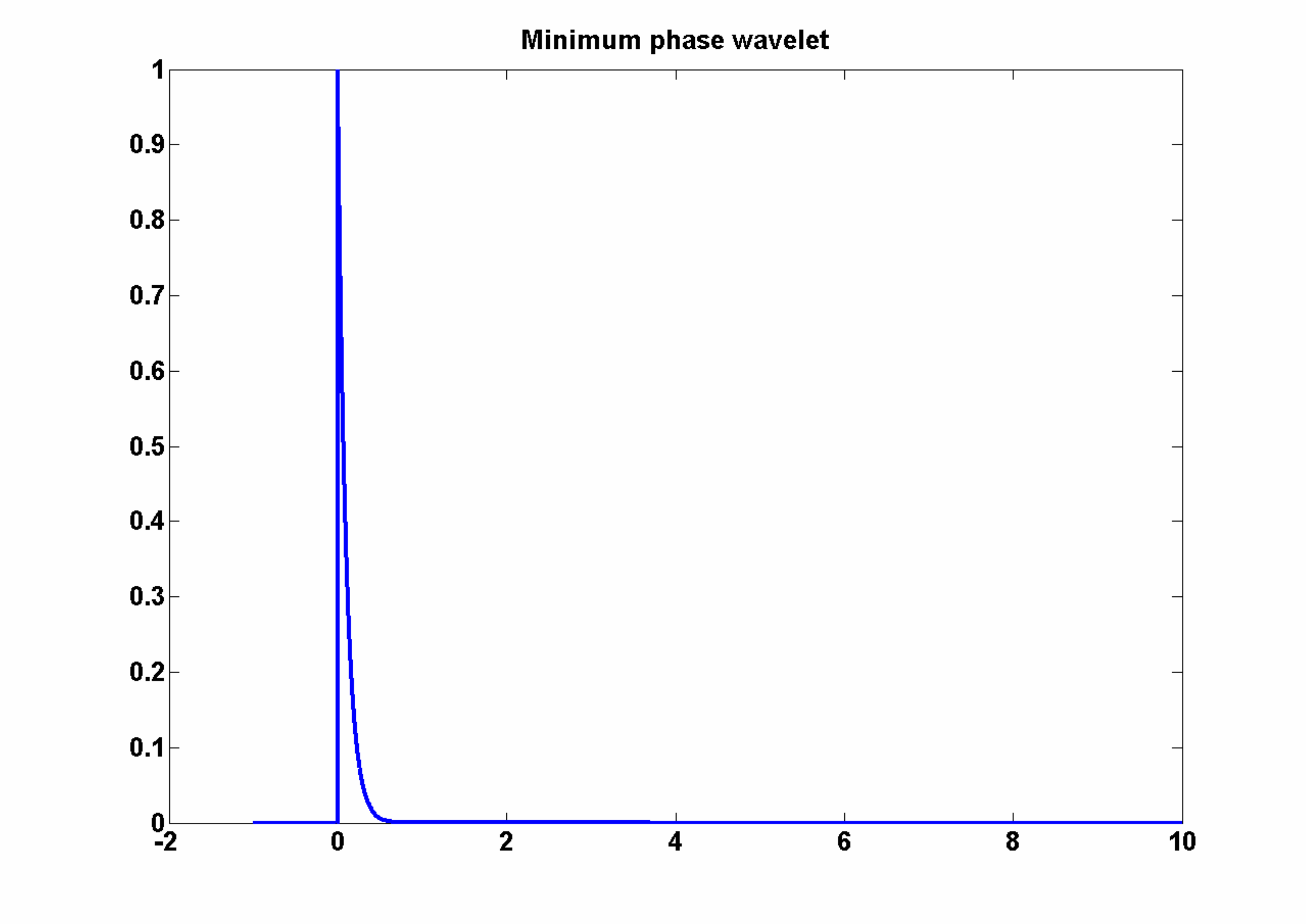
- has all the poles and zeros of its *Z*-transform inside the unit circle of the complex plane
- is causal, stable, and always has a minimum phase convolutional inverse
- has its energy concentrated toward time 0 more than any other causal signal having the same magnitude spectrum

Minimum phase for Digital systems

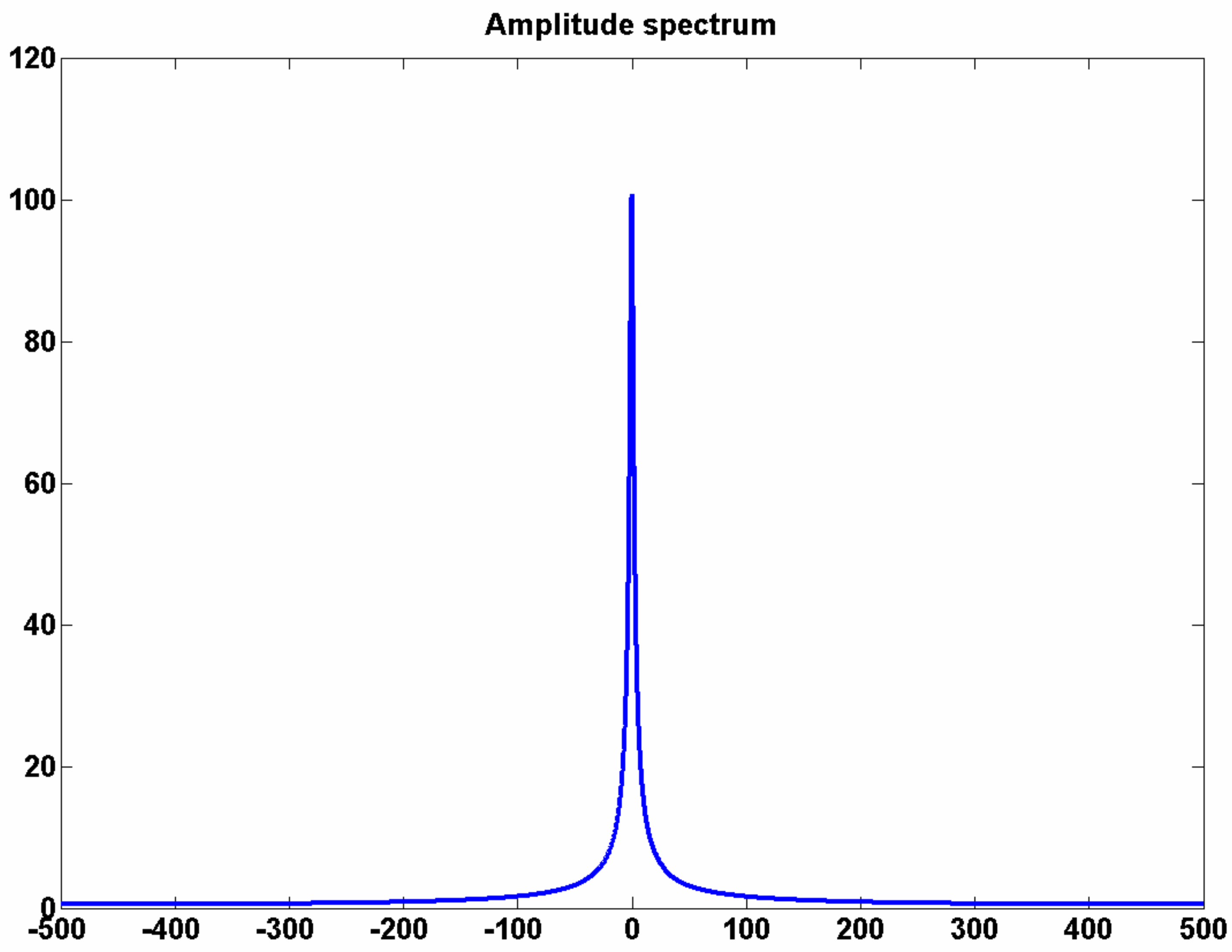
- Every causal, stable, all-pole digital filter is minimum phase
- Fermat's principle of least traveltime implies seismic wavelets are minimum phase
- Wiener and Gabor deconvolution assume that a constant-Q-attenuated seismic wavelet is minimum phase

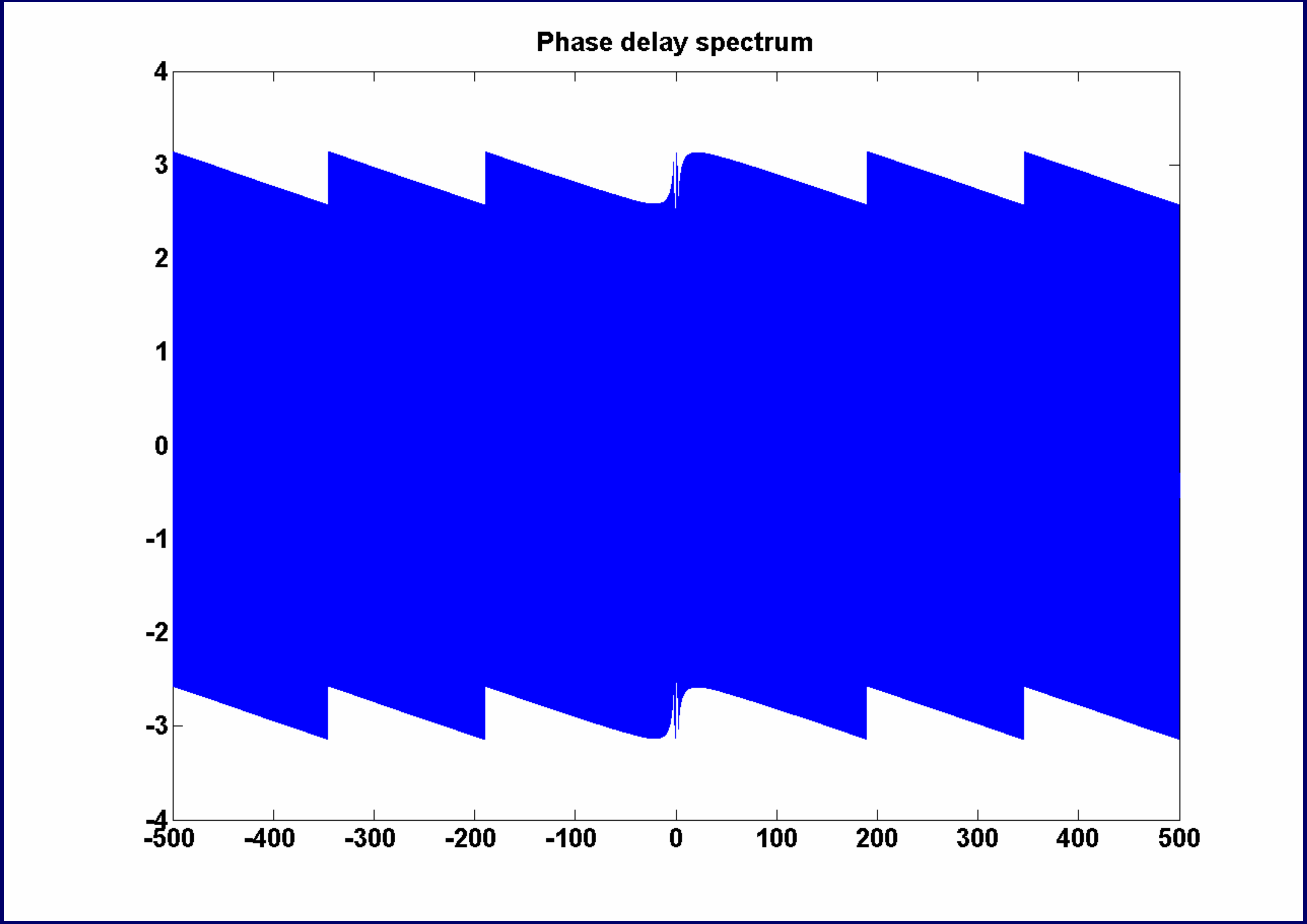
Fourier synthesis

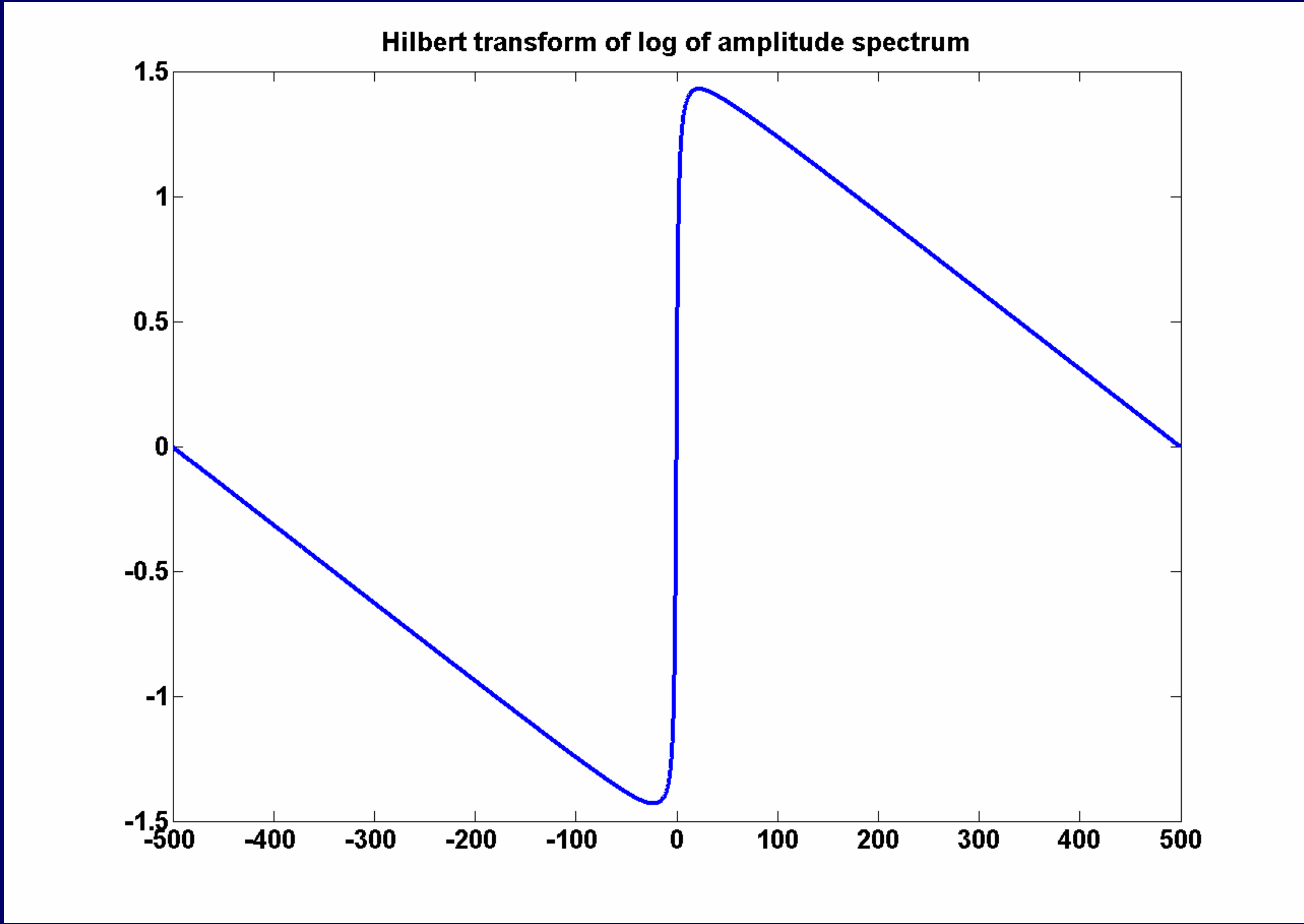
an example

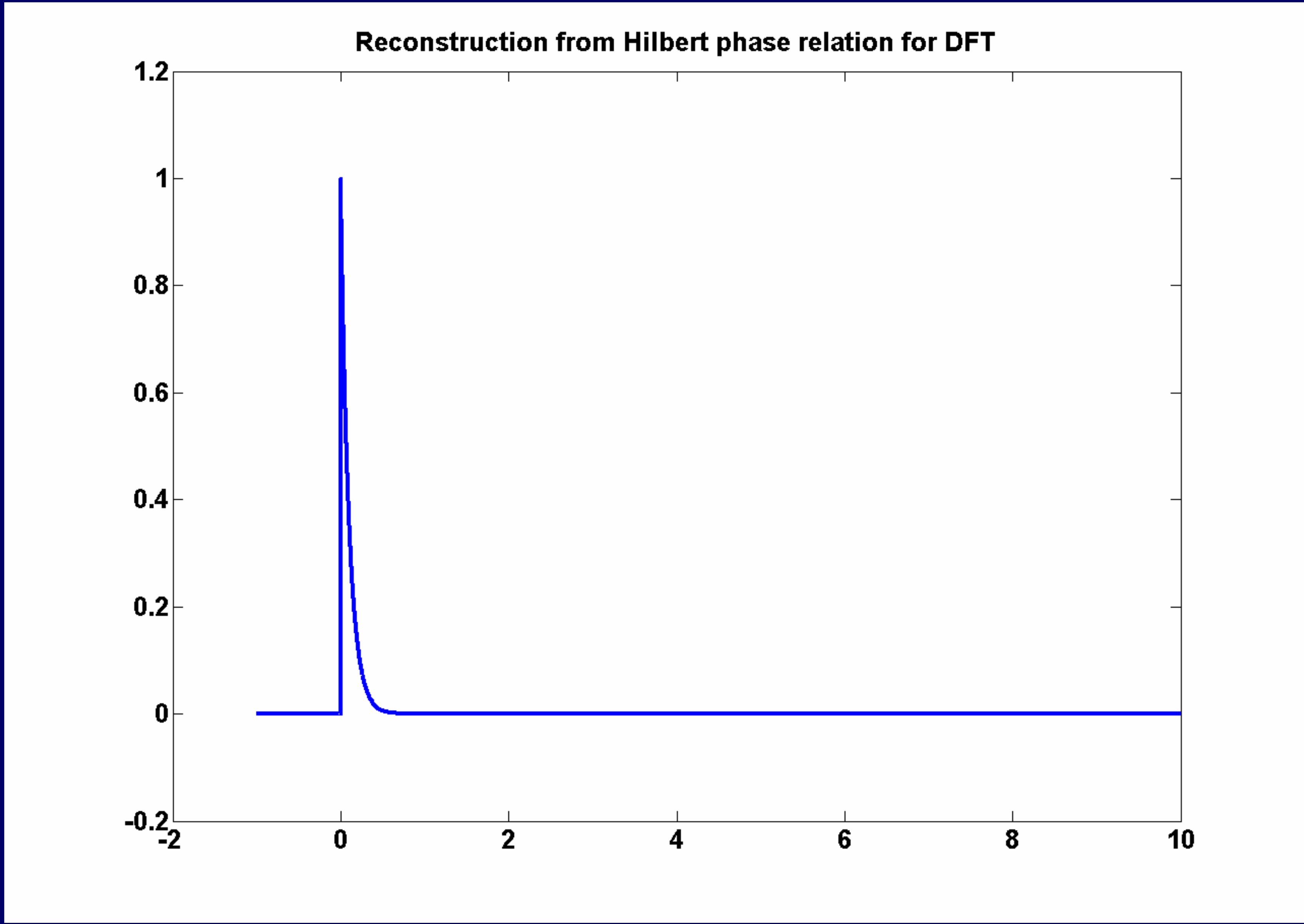


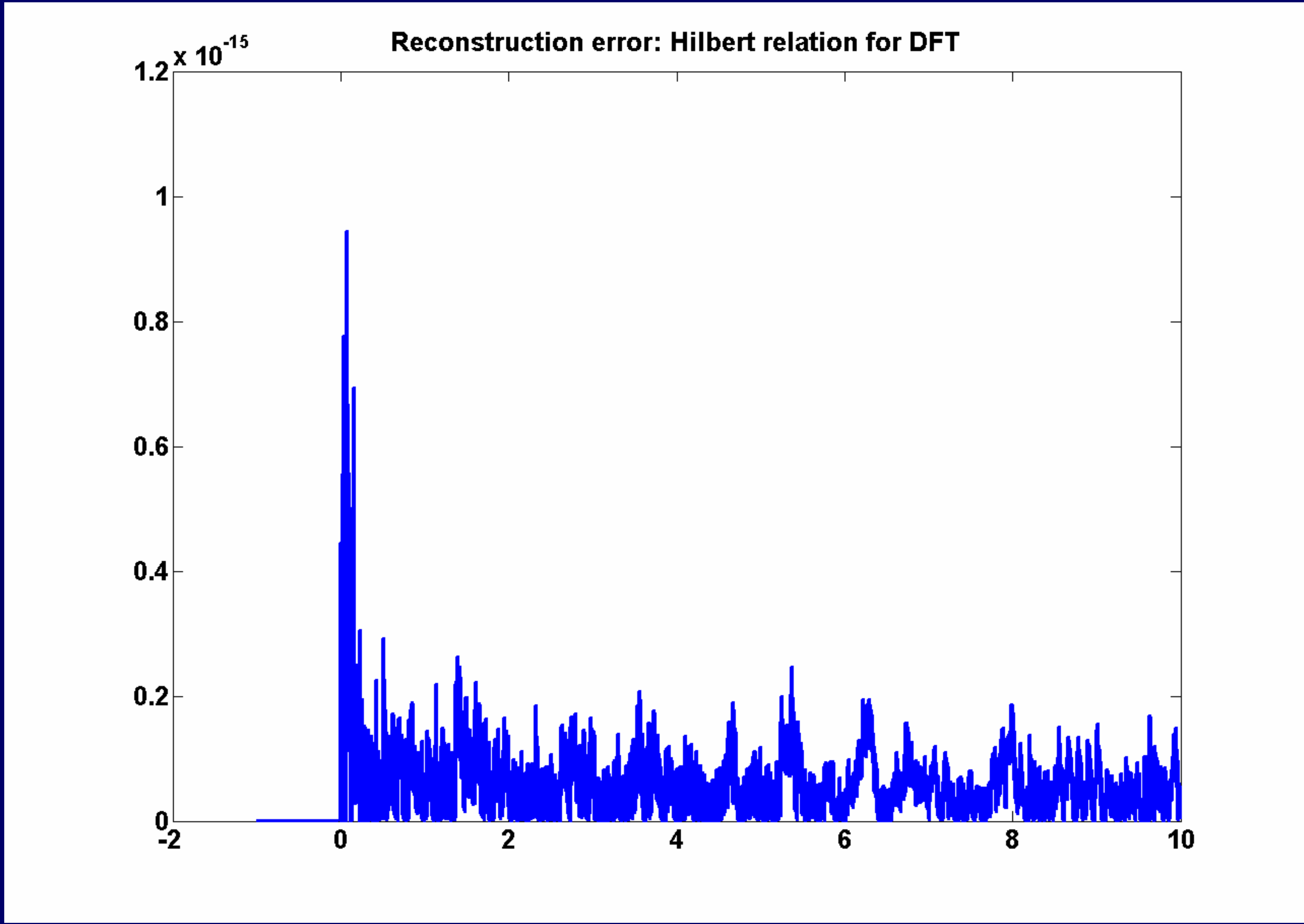
Minimum phase wavelet
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-2
0
2
4
6
8
10

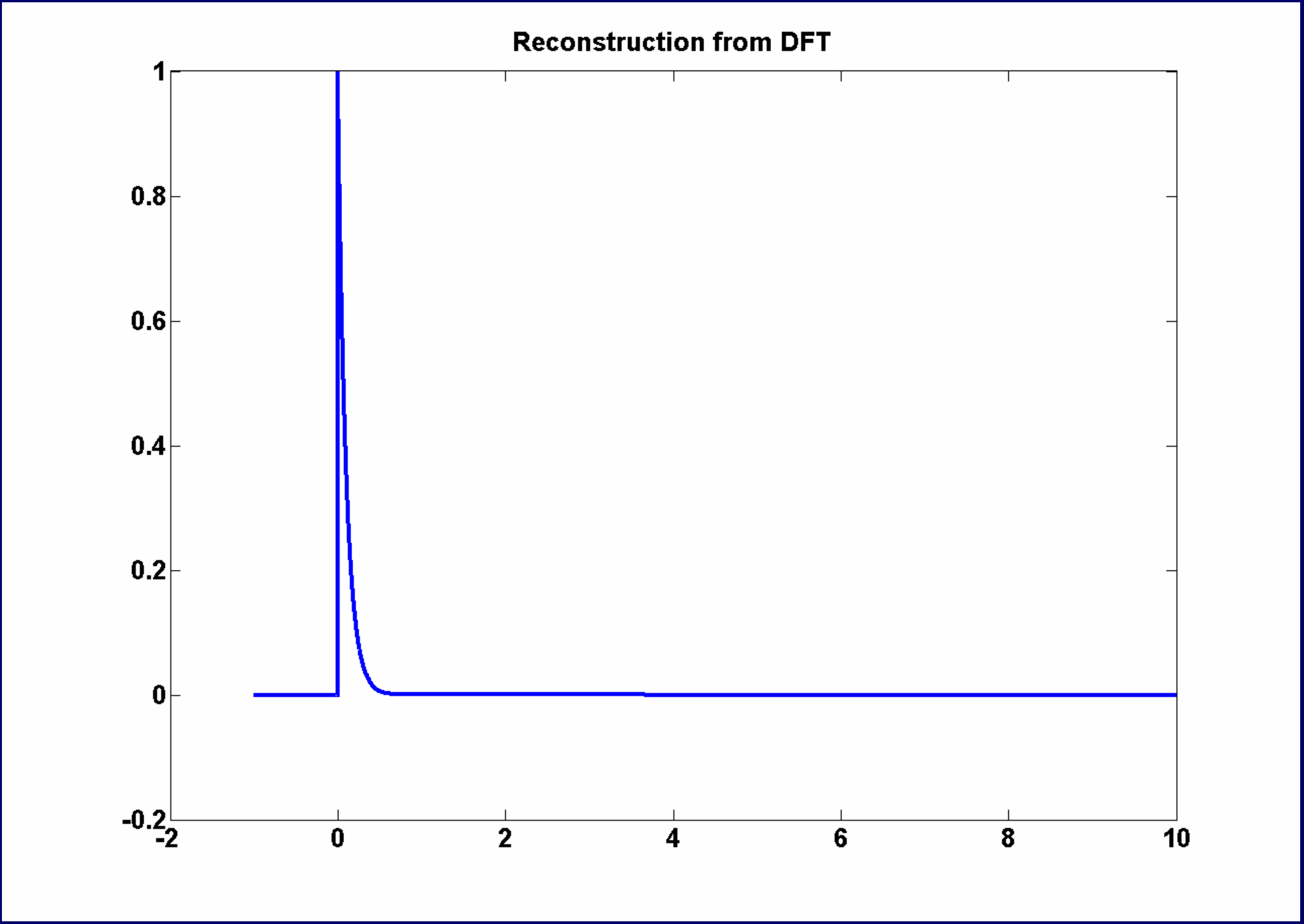

Amplitude spectrum
120
100
80
60
40
20
0
-500
-400
-300
-200
-100
0
100
200
300
400
500


Phase delay spectrum
4
3
2
1
0
-1
-2
-3
-4
-500
-400
-300
-200
-100
0
100
200
300
400
500

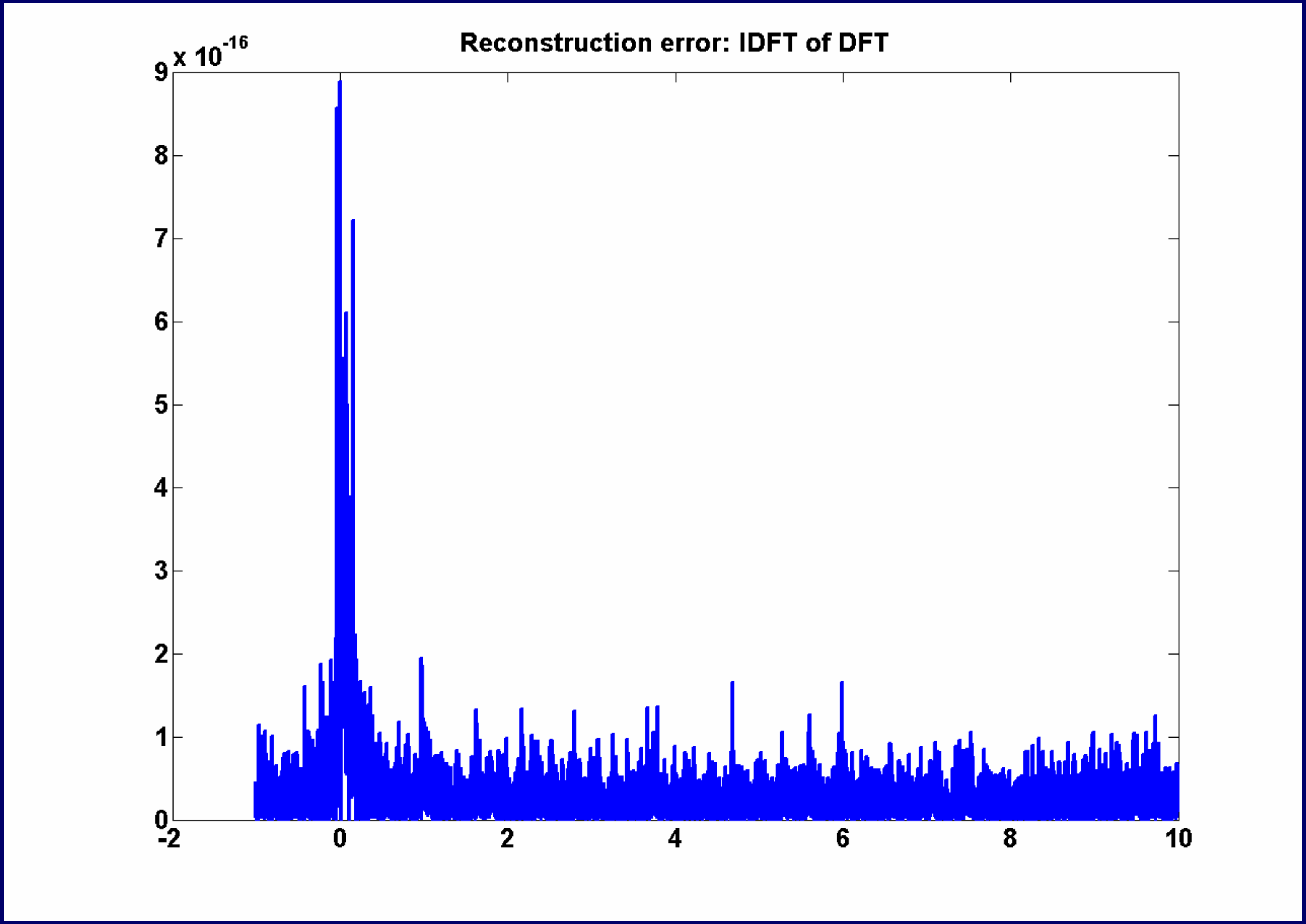

Hilbert transform of log of amplitude spectrum
1.5
1
0.5
0
-0.5
-1
-1.5
-500
-400
-300
-200
-100
0
100
200
300
400
500


Reconstruction from Hilbert phase relation for DFT
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-2
0
2
4
6
8
10


Reconstruction error: Hilbert relation for DFT
x 10^-15
1.2
1
0.8
0.6
0.4
0.2
0
-2
0
2
4
6
8
10

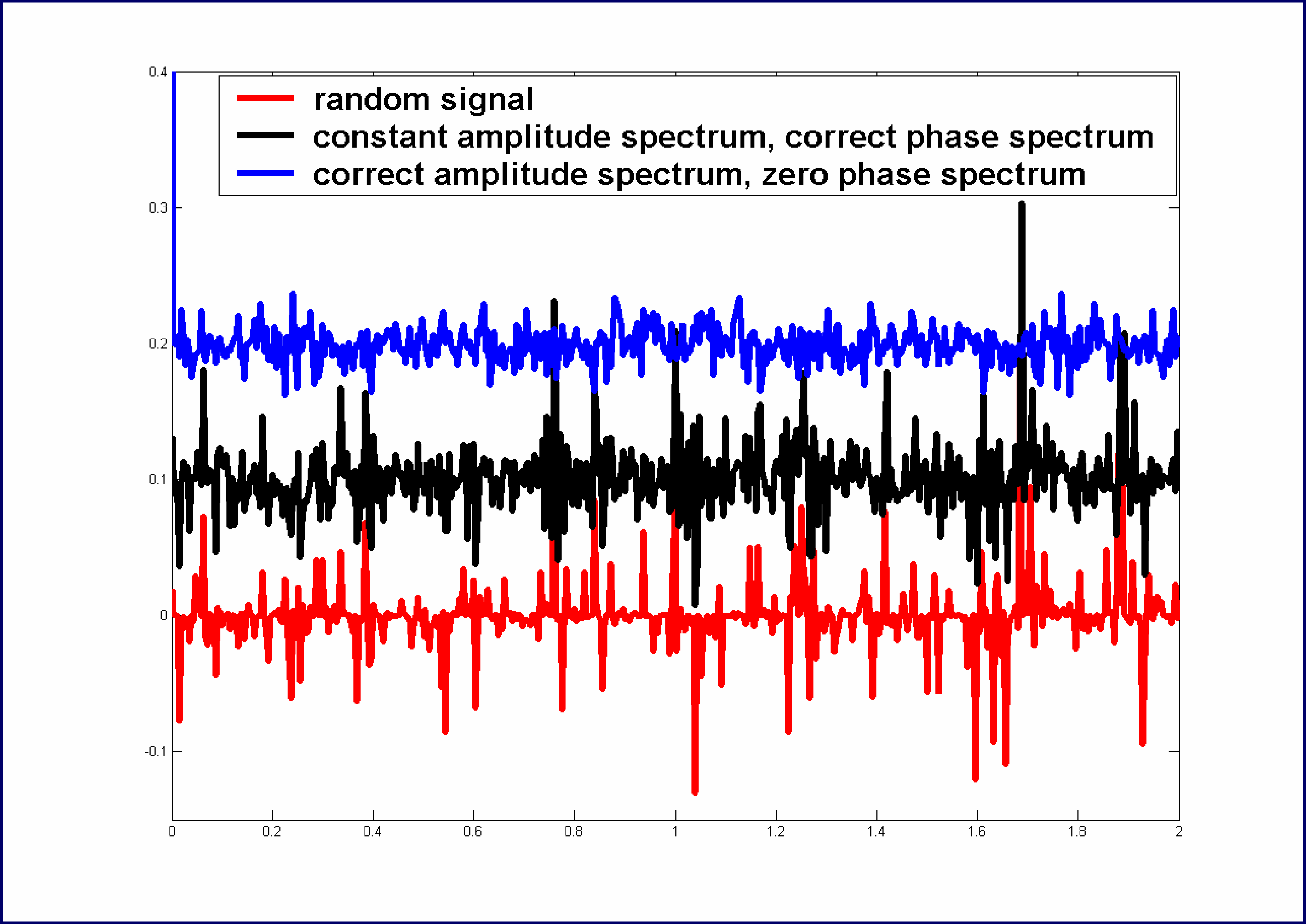


Reconstruction from DFT
1
0.8
0.6
0.4
0.2
0
-0.2
-2
0
2
4
6
8
10

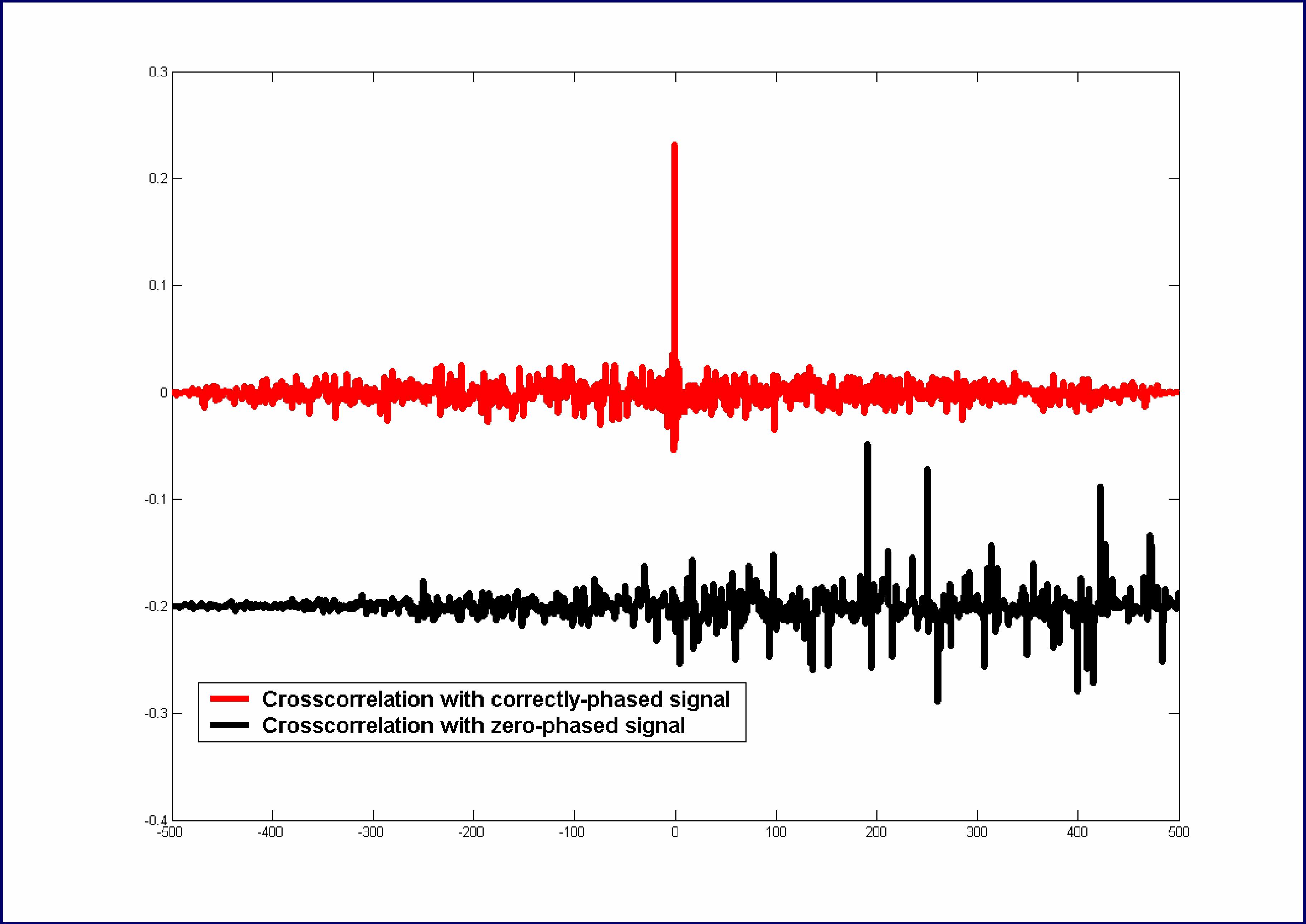

Reconstruction error: IDFT of DFT
x 10^-16
9
8
7
6
5
4
3
2
1
0
-2
0
2
4
6
8
10

Fourier synthesis for a Random spike series

Fourier synthesized signals



Crosscorrelations with original signal



Fourier Synthesis

- So, it seems phase accuracy is more important than amplitude spectrum accuracy…
- But, in Gabor/Wiener deconvolution, a phase spectrum is computed from an estimate of the amplitude spectrum

Tempered distributions

a class of generalized functions

Tempered distributions

- A way to make sense of divergent integrals
- A way to make sense of Dirac's delta "function"
- A tempered distribution is a continuous linear mapping T : S(**R**) → **C**
- S(**R**) is the Schwartz space of smooth, rapidly decreasing functions, e.g., Gaussians

Tempered distributions, some examples

Given a nice enough function *f* that doesn't grow too quickly,

$$T_f(\varphi) = \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx$$

defines a tempered distribution.

Tempered distributions, some examples

Any function *f* for which

$$\int_{-\infty}^{\infty} \left| f(x) \right|^p dx < \infty, \quad \text{for some } p \in [1, \infty)$$

defines a tempered distribution, T_f.

Tempered distributions, some examples

Dirac's delta function,

$$\delta(\varphi) = \varphi(0)$$

is a tempered distribution that doesn't correspond to any function.

Causality and the Hilbert transform

Causality and the Hilbert transform

Given a tempered distribution, T, write it as a sum of its causal and anti-causal parts:

$$T = T_+ + T_-,$$

Causality and the Hilbert transform

It turns out that the Hilbert transform is given simply as:

$$H\hat{T} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} \hat{T}_+ \\ \hat{T}_- \end{bmatrix} = i\hat{T}_+ - i\hat{T}_-$$

Minimum phase
for Analog systems

Minimum phase for Analog systems

A tempered distribution T is minimum phase if

$$\hat{T}, \quad 1/\hat{T}$$

correspond to analytic functions in the lower half of the complex plane,

$$C_{-} = \{z \in C: \ \mathrm{imag}(z) < 0\}.$$

Examples

δ is minimum phase:

$$\hat{\delta} = 1,$$

which is analytic everywhere.

Also, $H(\ln \hat{\delta}) = 0,$

which happens to correspond to the phase spectrum

Examples

$f(x) = h(x)e^{-ax}$, for any $a > 0$:

$$\hat{f}(\xi) = \frac{1}{a + i\xi}, \quad \frac{1}{\hat{f}(\xi)} = a + i\xi,$$

analytic everywhere except at $\xi = ai$.

The phase spectrum is NOT given by

the Hilbert transform of $\ln\left|\hat{f}\right|$.

Why not?

It turns out that for T minimum phase,

$$\varphi_{\mathrm{T}} = H(\ln|\hat{T}|)$$

only if

$$\ln \hat{T} \to 0$$

along every ray in C_-.

Examples

If s is a causal Schwartz function, so is $\hat{s}$, and $e^{\hat{s}}$ corresponds to a minimum phase tempered distribution.

The phase spectrum agrees with the Hilbert transform relationship since $\hat{s}$ is causal.

Summary

- In the analog examples, $\hat{f}$ was invertible, but $1/\hat{f}$ was not "stable" (not integrable, or it had infinite energy).
- In each case, a convolution could reduce the original distribution to a spike, by division in the Fourier domain:

$$\hat{f}\,\frac{1}{\hat{f}} = 1,$$

which is the Fourier transform of δ.

Summary

- Sometimes the Hilbert transform relation fails for analog minimum phase signals, but it works very well on bandlimited versions.
- We computed a simpler min phase spectrum than the original ringy one, yet reconstructed the minimum phase signal with very high fidelity.

Acknowledgements

- The CREWES Project and its sponsors
- The POTSI Project and MITACS
- My co-authors/supervisors
- My energetic son, for waking me up at night so I could think about wavelets.