The annular-sum and Hale-McClellan methods of 3D wavefield extrapolation

Gary Margrave, Saleh Al Saleh,

Michael Lamoureux, Wentao Sun

1. Hale-McClellan, annular sums, background
2. Properties of annular-sum filters
3. Examples of annular-sum filters
4. Comparison with McClellan filters
5. Posing the least-squares (Wiener filter) problem for radial filters

Hale-McClellan

3D wavefield extrapolation can be done by the formula

$$\psi(\boldsymbol{x}, z+\Delta z, \omega) \approx w_0 \psi(\boldsymbol{x}, z, \omega) + w_1 (m_1 \bullet \psi)(\boldsymbol{x}, z, \omega) + \cdots$$

$$+ w_N (m_N \bullet \psi)(\boldsymbol{x}, z, \omega)$$

$\psi(\boldsymbol{x}, z, \omega) \ldots$ space-frequency domain wavefield

$w_j \ldots$ samples of 2D wavefield extrapolator

$m_j \ldots j^{\text{th}}$ McClellan filter

$\bullet \ldots$ 2D convolution over $\boldsymbol{x} = (x_1, x_2)$

Annular sums

3D wavefield extrapolation can be done by the formula

$$\psi(\boldsymbol{x}, z+\Delta z, \omega) \approx W_0 (a_0 \bullet \psi)(\boldsymbol{x}, z, \omega) + W_1 (a_1 \bullet \psi)(\boldsymbol{x}, z, \omega) + \cdots + W_N (a_N \bullet \psi)(\boldsymbol{x}, z, \omega)$$

$\psi(\boldsymbol{x}, z, \omega) \ldots$ space-frequency domain wavefield

$W_j \ldots$ radial samples of 3D wavefield extrapolator

$a_j \ldots j^{\text{th}}$ annular-sum filter

$\bullet \ldots$ 2D convolution over $\boldsymbol{x} = (x_1, x_2)$

Background facts

about space-frequency extrapolators

2D

2D wavefield extrapolation is a 1D spatial convolution at constant frequency.

2D wavefield extrapolator is a complex-valued symmetric (even) function in space.

$$w(\rho_1, k, \Delta z) \propto$$

$$\frac{\sqrt{k}\Delta z}{\rho_1^{3/2}(1+\ldots)} e^{i\left[k\sqrt{\rho_1^2+\Delta z^2}-\pi/4\right]}$$

3D

3D wavefield extrapolation is a 2D spatial convolution at constant frequency.

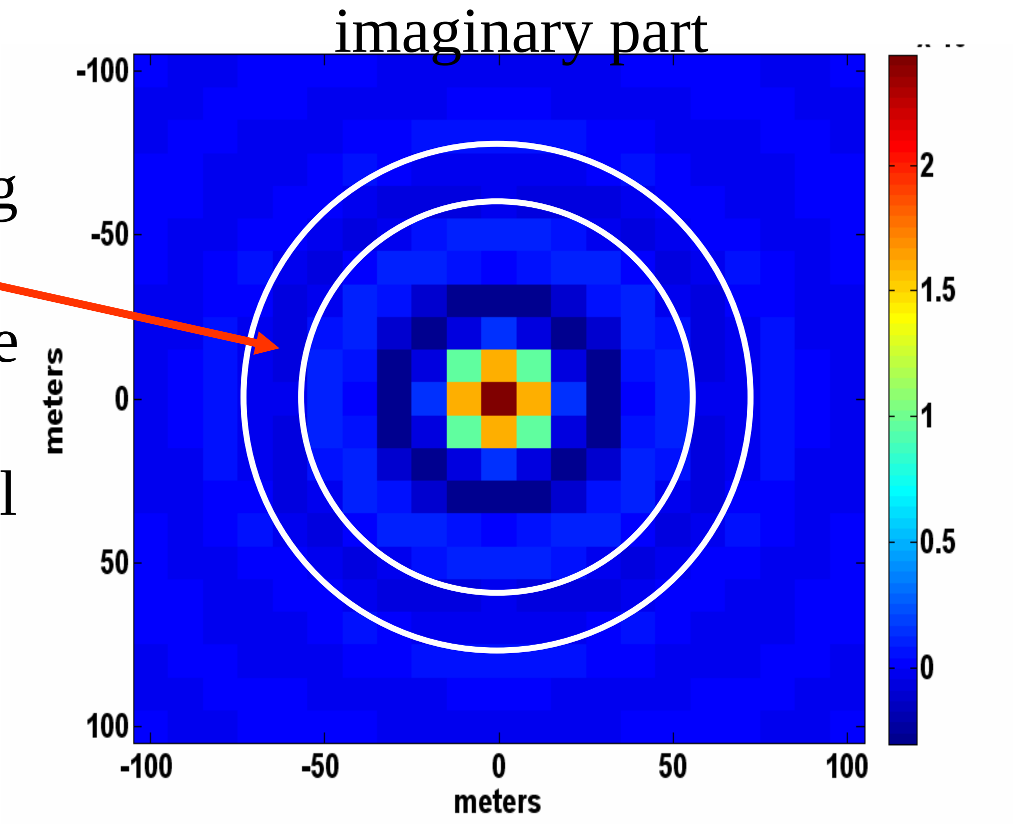
3D wavefield extrapolator depends only on radius, i.e. is circularly symmetric.

$$W(\rho, k, \Delta z) \propto$$

$$\frac{k\Delta z}{\rho^2(1+\ldots)} e^{i\left[k\sqrt{\rho^2+\Delta z^2}-\pi/2\right]}$$

3D Wavefield Extrapolator

All wavefield samples falling in a 1-sample annulus will be multiplied by the same radial sample of the operator.



The output from an annular-sum filter is a new field whose value at any point is the sum of the values contained in an annulus centered at that point in the input field.

Annular-sum filter

3 methods

1) $\bar{\psi}_n(\boldsymbol{x}, z, \omega) = \left(I_{\rho_{n+1}} - I_{\rho_n}\right) \bullet \psi(\boldsymbol{x}, z, \omega)$ Direct convolution

$I_{\rho_n}(\boldsymbol{x}) = \begin{cases} 1, |\boldsymbol{x}| \le \rho_n \\ 0, \text{otherwise} \end{cases}, n > 0; \quad I_{\rho_0} = 0$ Indicator function of a disk

2) $\bar{\psi}_n(\boldsymbol{x}, z, \omega) = \left(F_2^{-1}\left(\hat{I}_{(n+1)\Delta x} - \hat{I}_{n\Delta x}\right) F_2 \psi\right)(\boldsymbol{x}, z, \omega)$ Disk difference

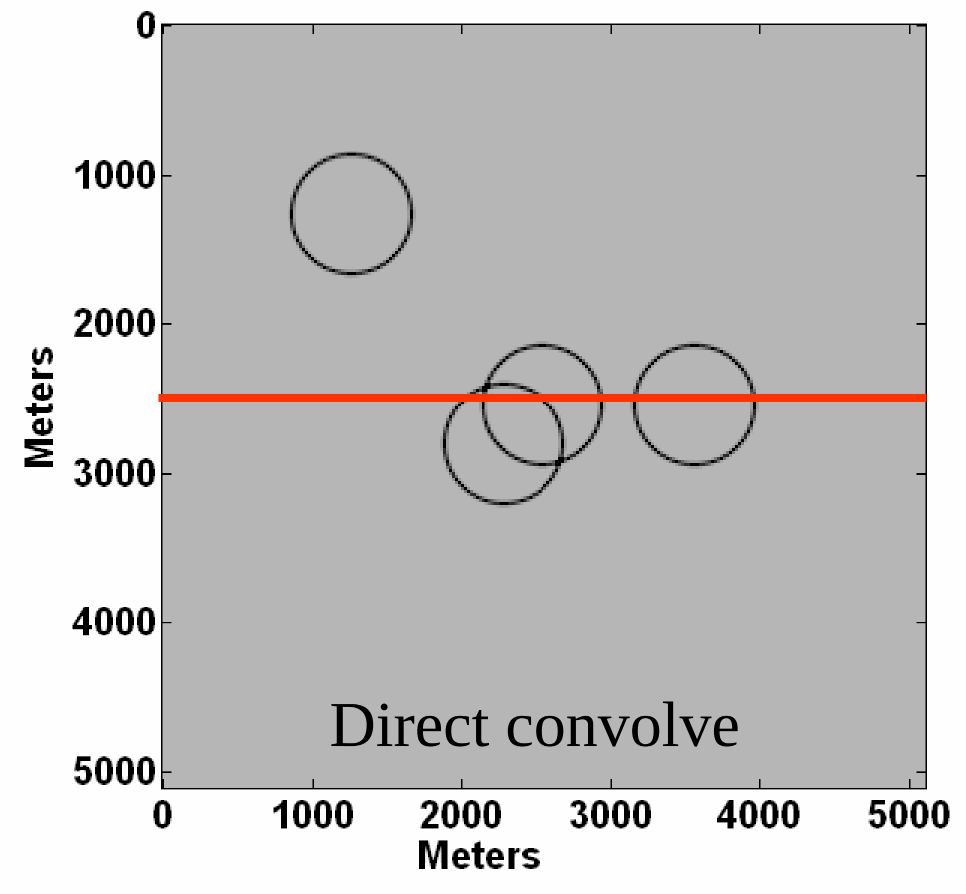
$\hat{I}_\rho(|\boldsymbol{\xi}|) = \frac{\rho J_1(2\pi\rho|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}$ Fourier transform of disk indicator function.

3) $\bar{\psi}_n(\boldsymbol{x}, z, \omega) \approx \Delta x F_2^{-1}\left(\hat{I}_{\delta(n\Delta x)} F_2 \psi\right)(\boldsymbol{x}, z, \omega)$ Delta annulus

$\hat{I}_{\delta\rho}(|\boldsymbol{\xi}|) = 2\pi\rho J_0(2\pi\rho|\boldsymbol{\xi}|)$ Fourier transform of radial delta function

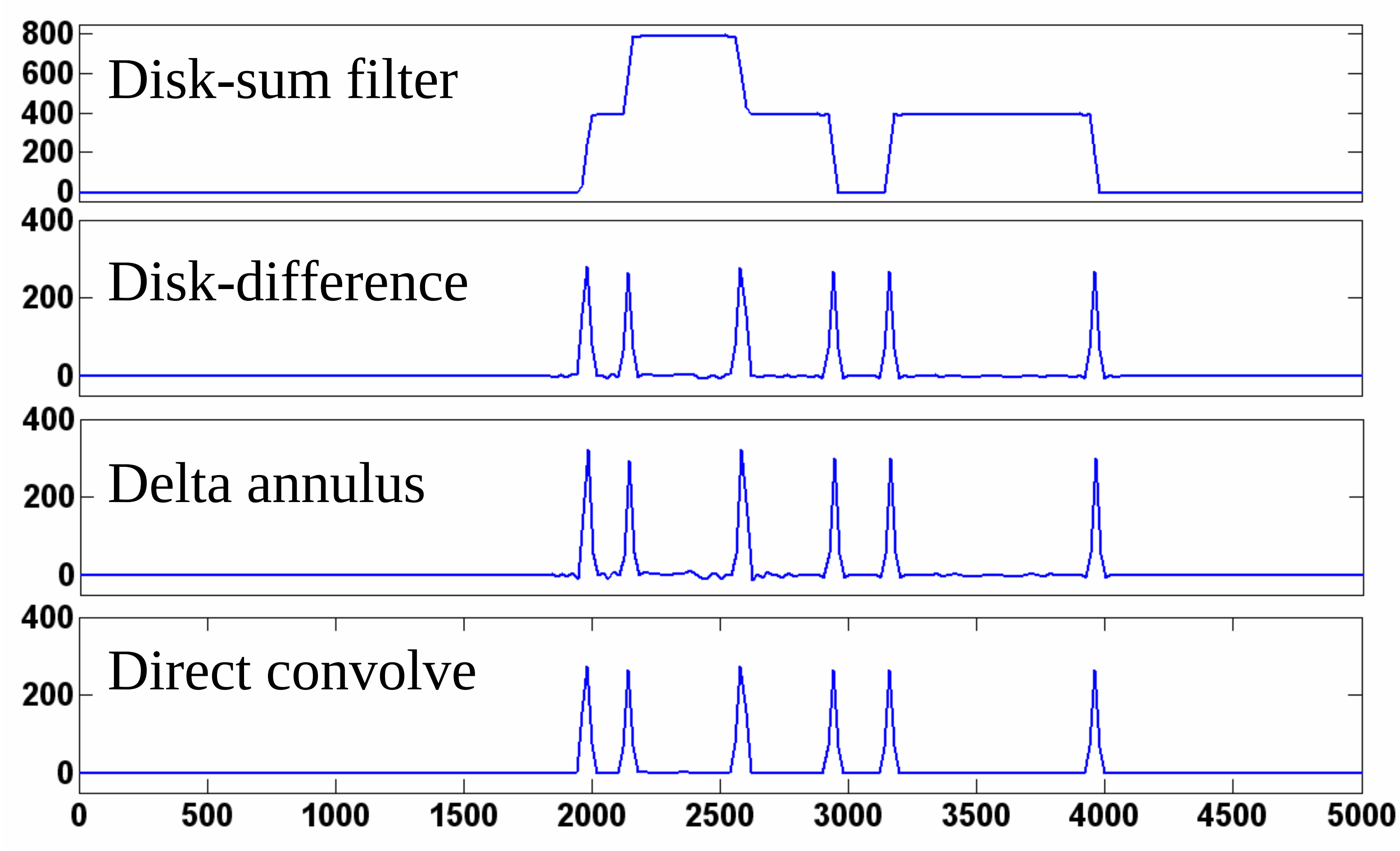
Annular-sum filter

example



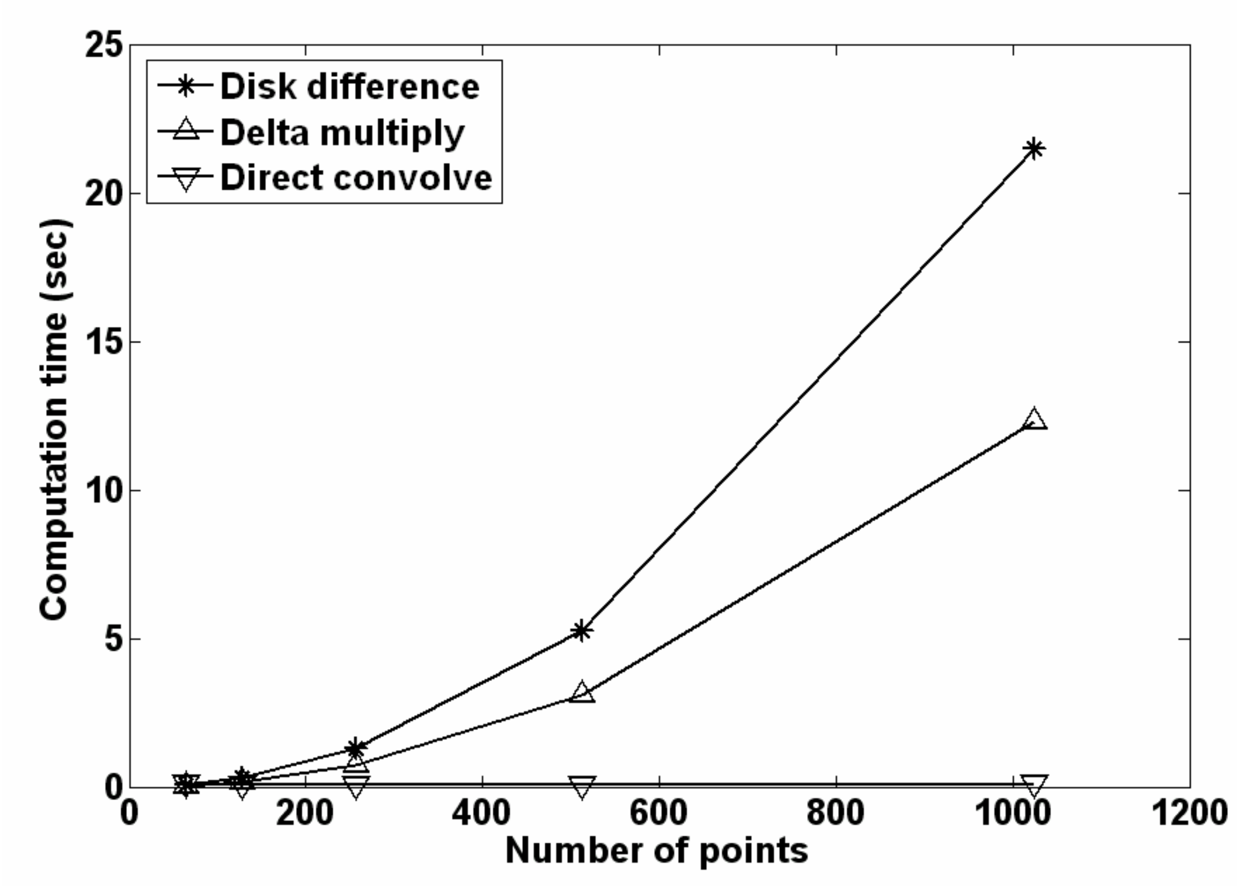
Annular-sum filter

example



Annular-sum filter

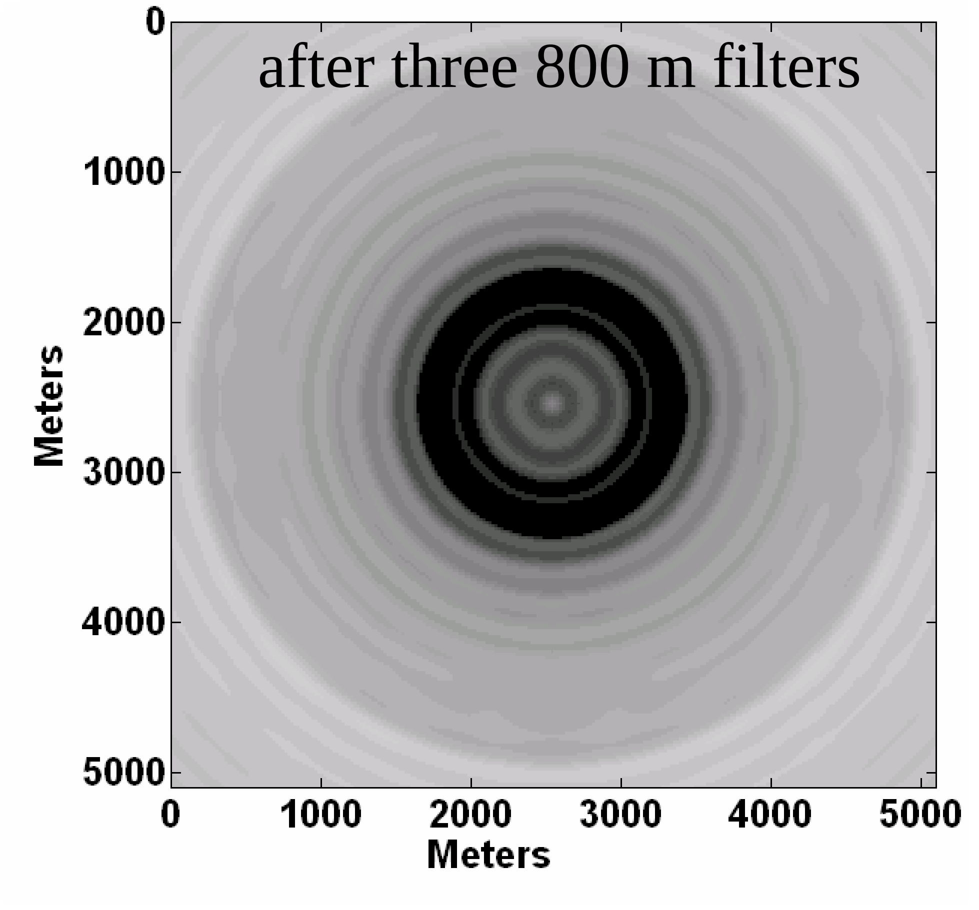
cost



The Fourier methods scale as NlogN while the direct convolve is much faster.

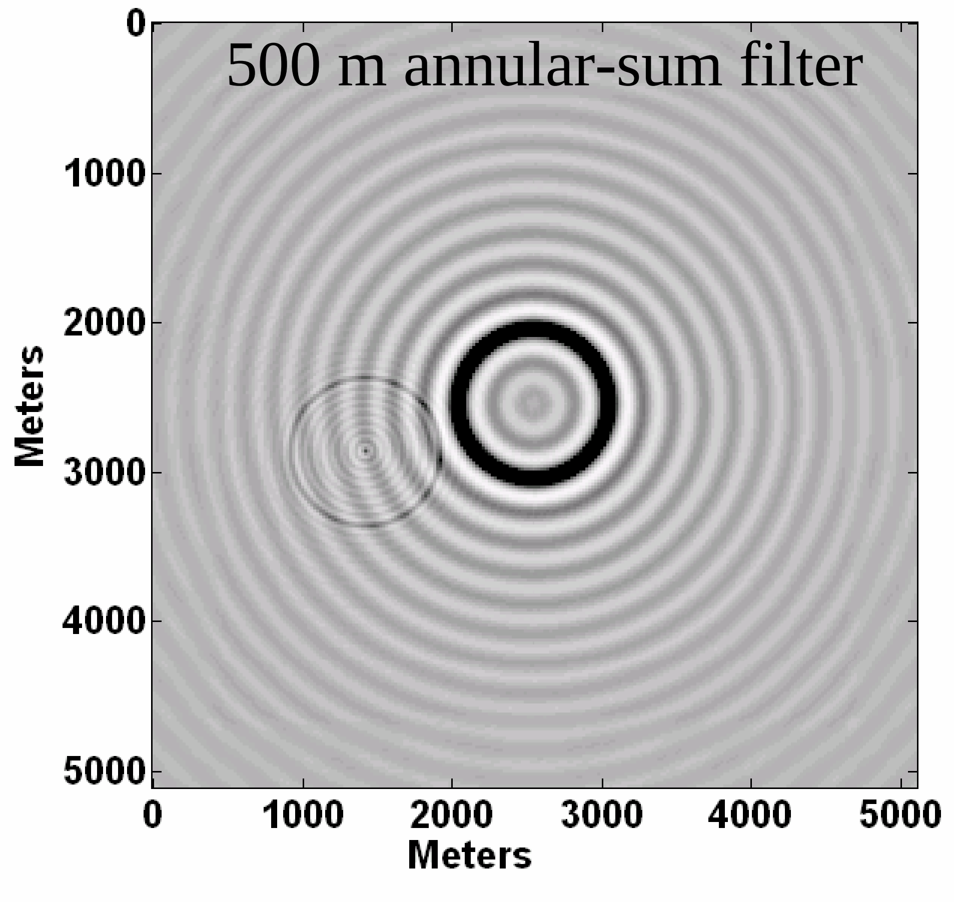
Annular-sum filter

second example



Annular-sum filter

third example



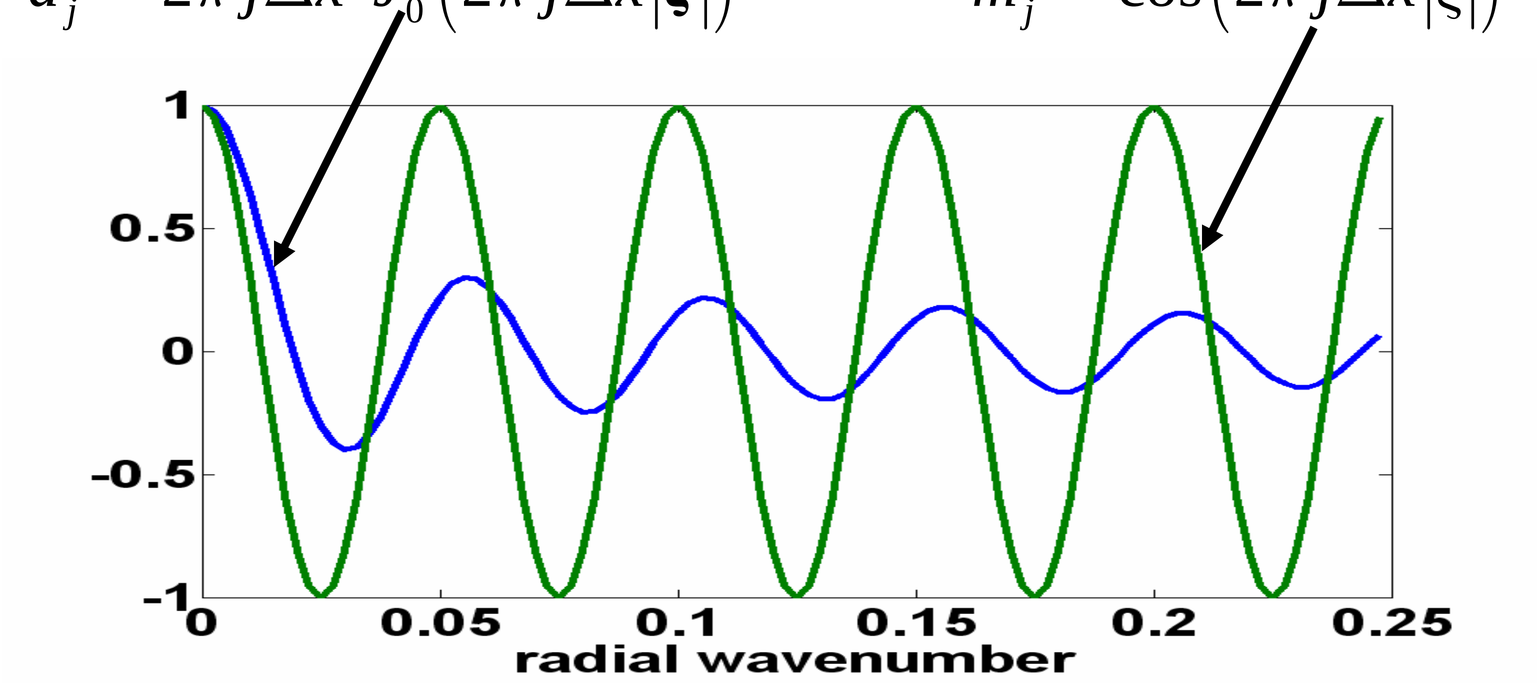
Annular-sum and McClellan filters
in Fourier domain

Annular sum

$$\hat{a}_j = 2\pi j \Delta x^2 J_0\left(2\pi j \Delta x |\xi|\right)$$

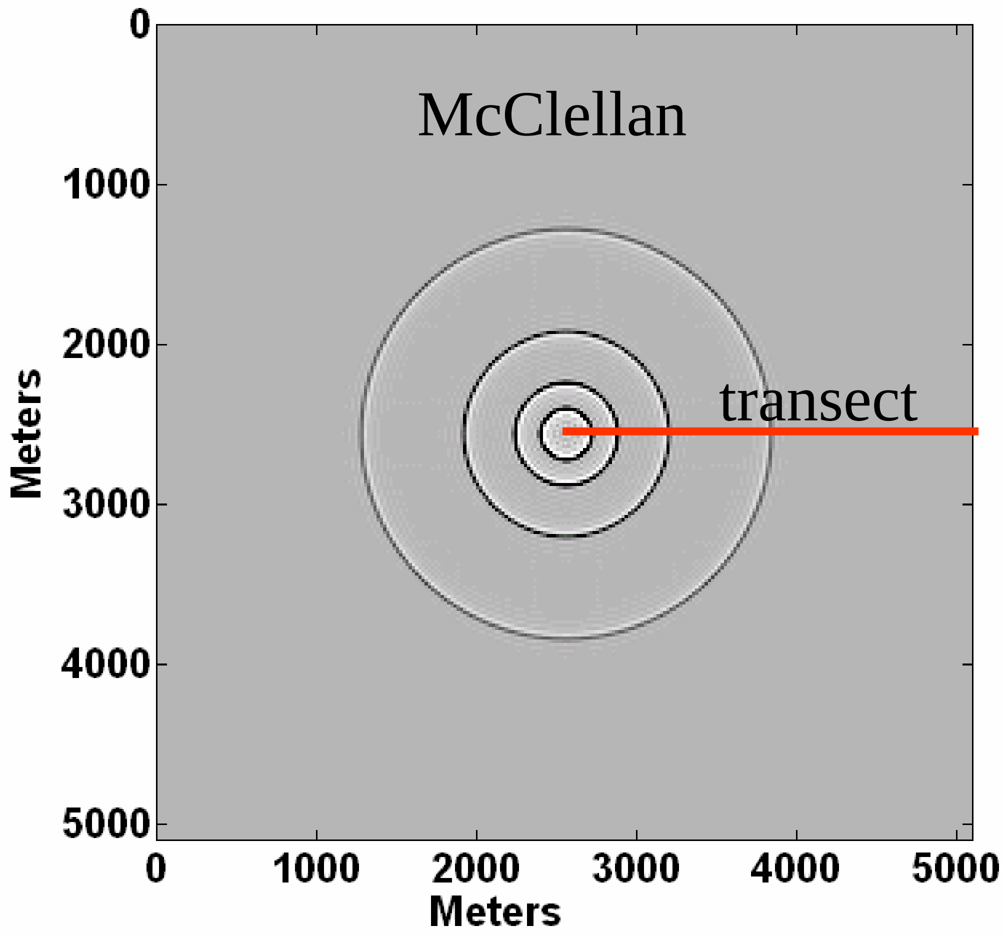
McClellan

$$\hat{m}_j = \cos\left(2\pi j \Delta x |\xi|\right)$$



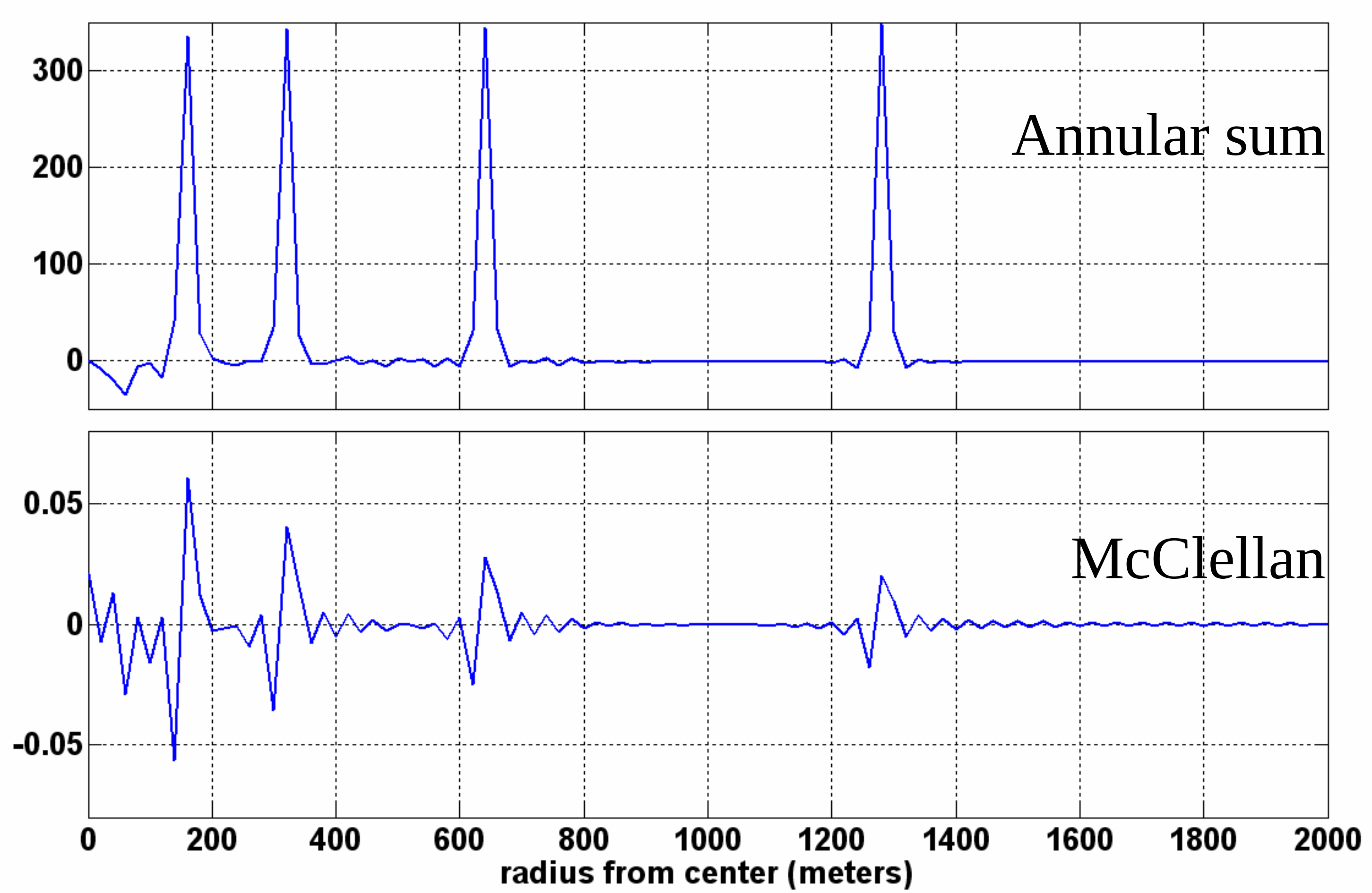
Annular-sum and McClellan filters

in space domain



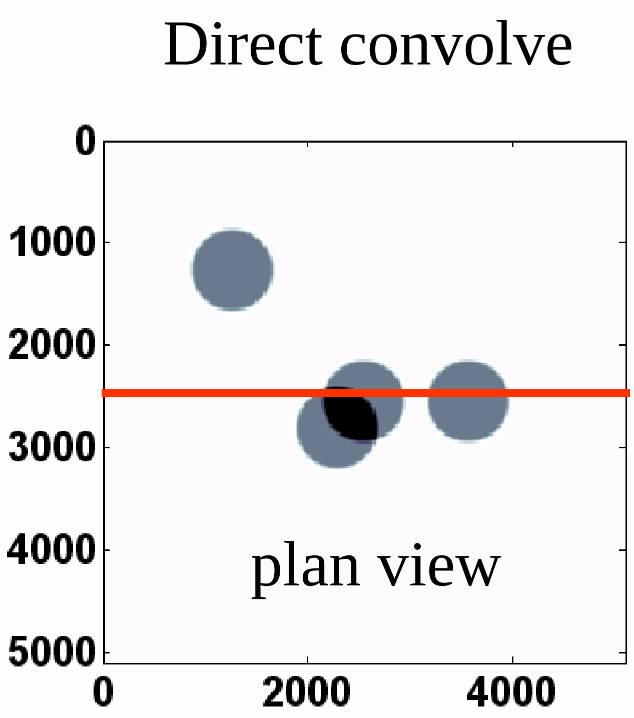
Annular-sum and McClellan filters

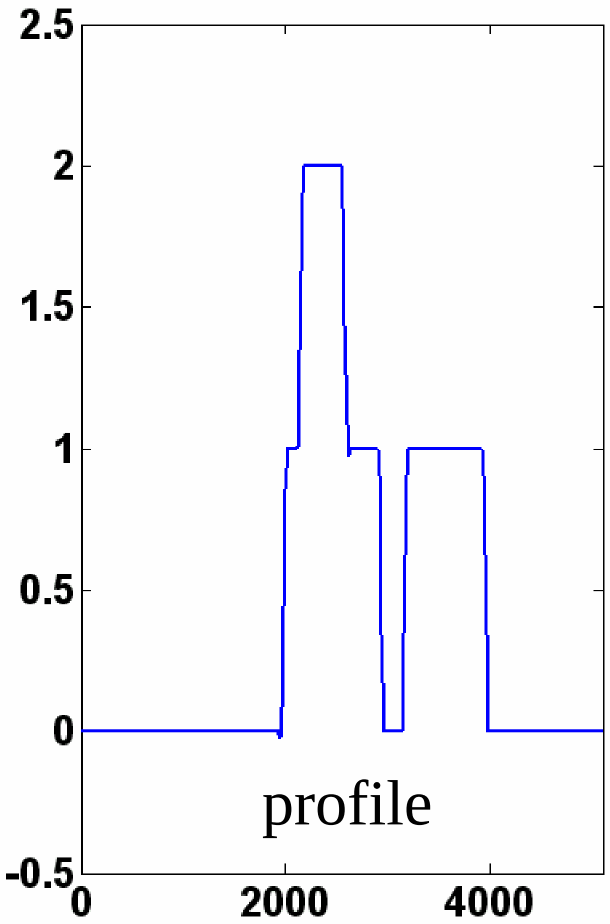
in space domain



McClellan filters have the extra decay and phase rotation required to convert the 2D extrapolator into a 3D extrapolator.

Disk convolution via annular-sums





Designing the 3D operator

two possibilities

1. Design a 2D operator and pass it through the McClellan transform

2. Design the 3D operator directly by least squares

Radial convolution matrix

enables least-squares for radial operators

$W(\rho) \bullet V(\rho) = A(\rho)$ 2D convolution of radial operators

equivalent matrix equation

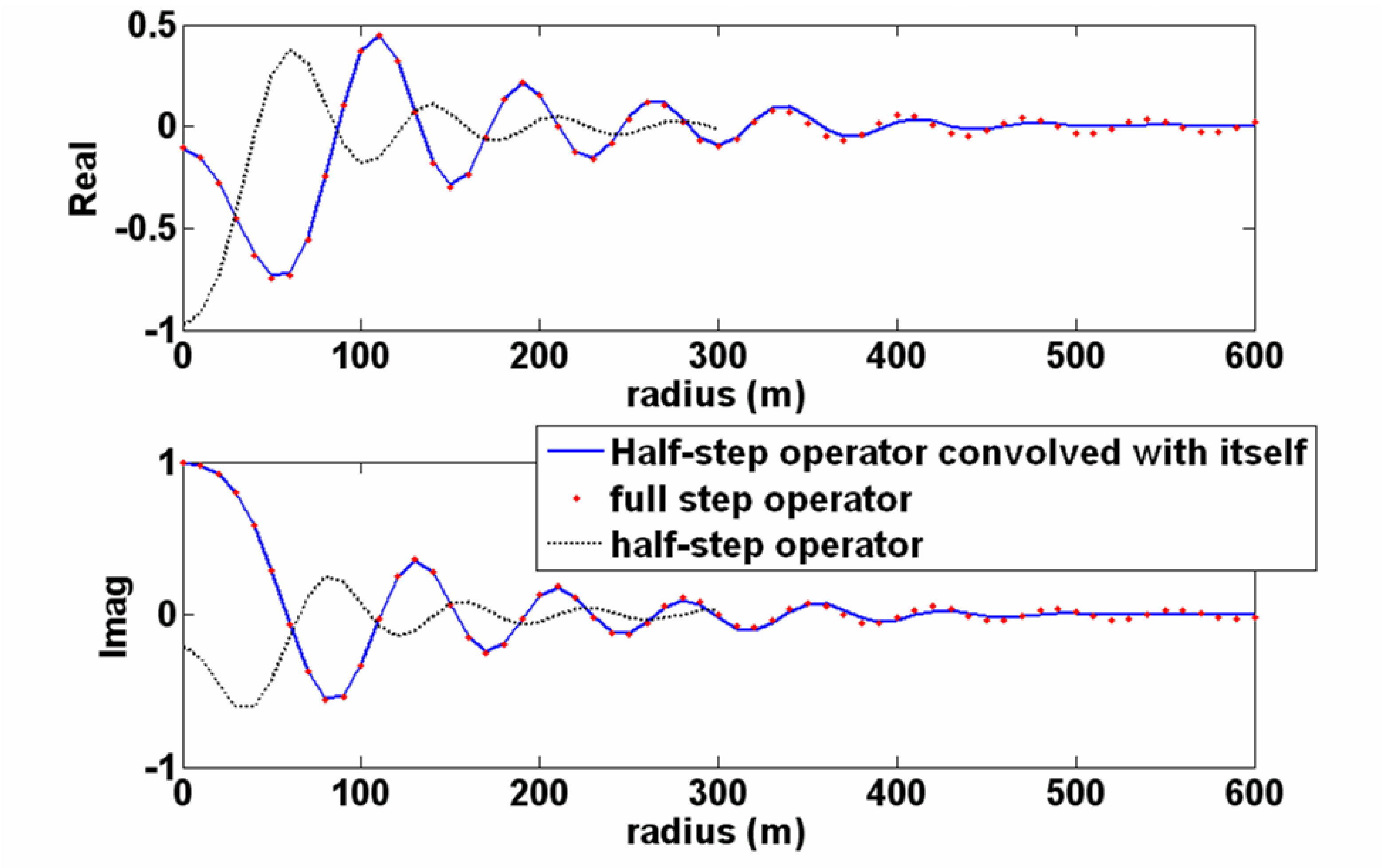
$$\begin{bmatrix} \bar{W}_{00} & \bar{W}_{01} & \cdots & \bar{W}_{0M} \\ \bar{W}_{10} & \bar{W}_{11} & \cdots & \bar{W}_{1M} \\ \bar{W}_{20} & \bar{W}_{21} & \cdots & \bar{W}_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{W}_{N0} & \bar{W}_{N1} & \cdots & \bar{W}_{NM} \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_M \end{bmatrix} = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \\ A_N \end{bmatrix}$$

radial samples of *A*

matrix of annular sums computed from the 2D impulse response of *W*

radial samples of *V*

Example using the radial convolution matrix



Conclusions

- Annular-sum filters facilitate a 3D wavefield extrapolation formula comparable to Hale-McClellan
- Annular-sum filters can be implemented with fidelity and efficiency
- McClellan filters are a type of annular-sum filter that includes extra decay and phase rotation
- 3D wavefield extrapolator design by least-squares appears feasible using the radial convolution matrix

Acknowledgements

We thank the industrial sponsors of CREWES and POTSI.

We also gratefully acknowledge the assistance of MITACS, PIMS, and NSERC.