Differential operators

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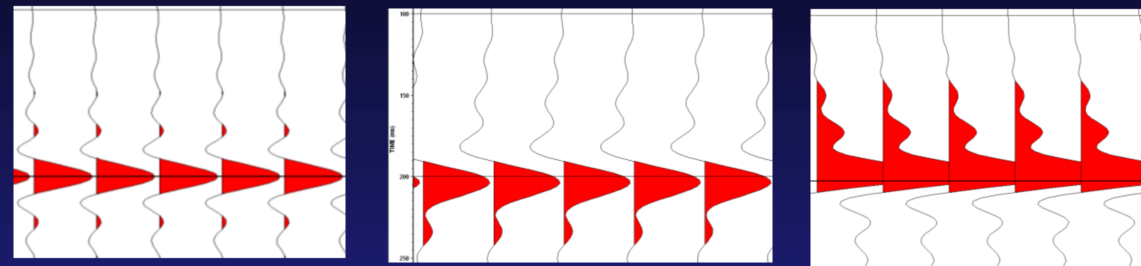
Elastic **W**ave **E**xploration **S**eismology

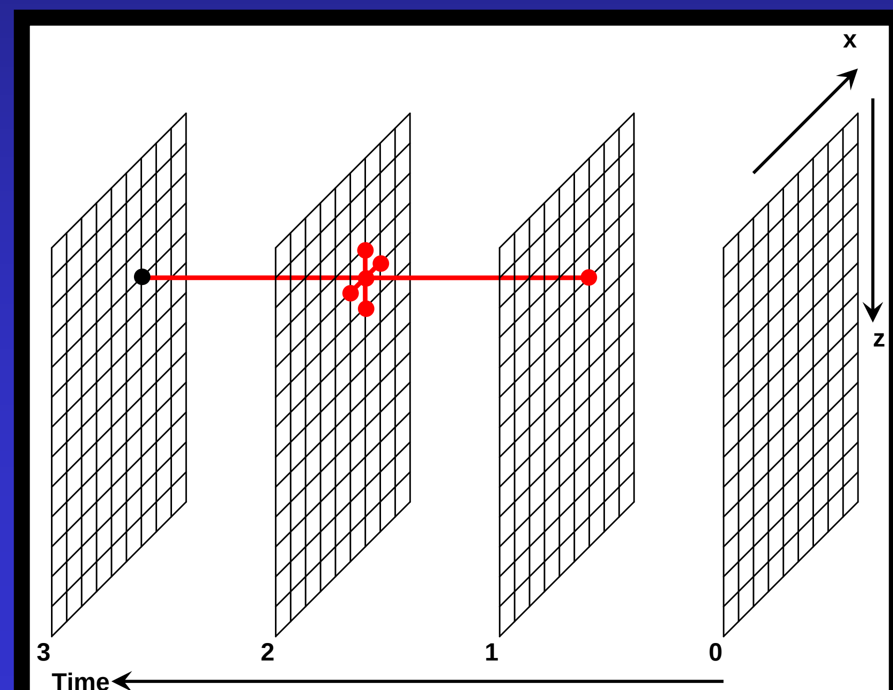
NSERC

Motivation:

- Modelling and migration

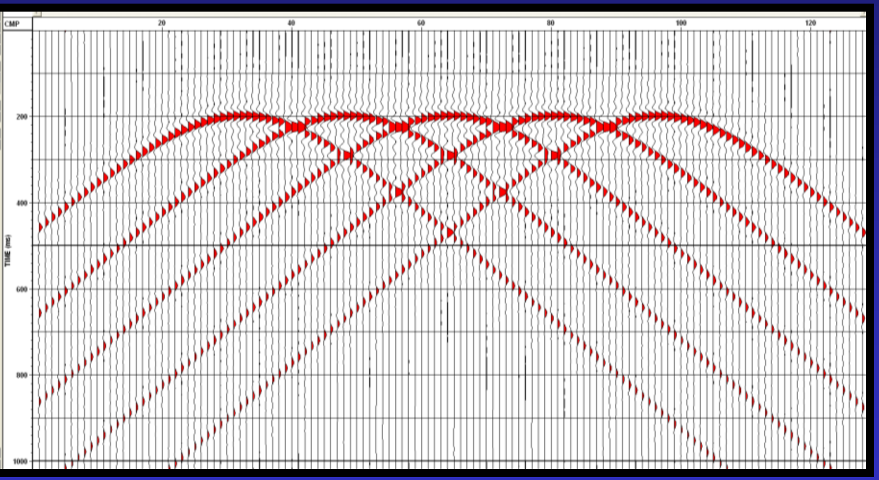


- Solving the wave-equation



Modelling wavelet

- Start with a diffraction and zero-phase wavelet
- Build a reflection



Modelling wavelet

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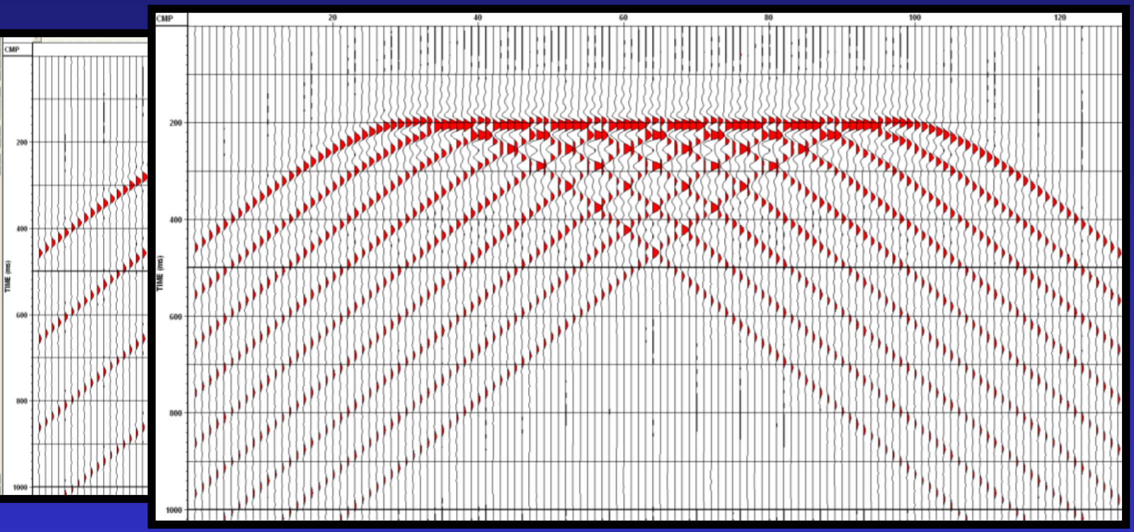

Modelling wavelet

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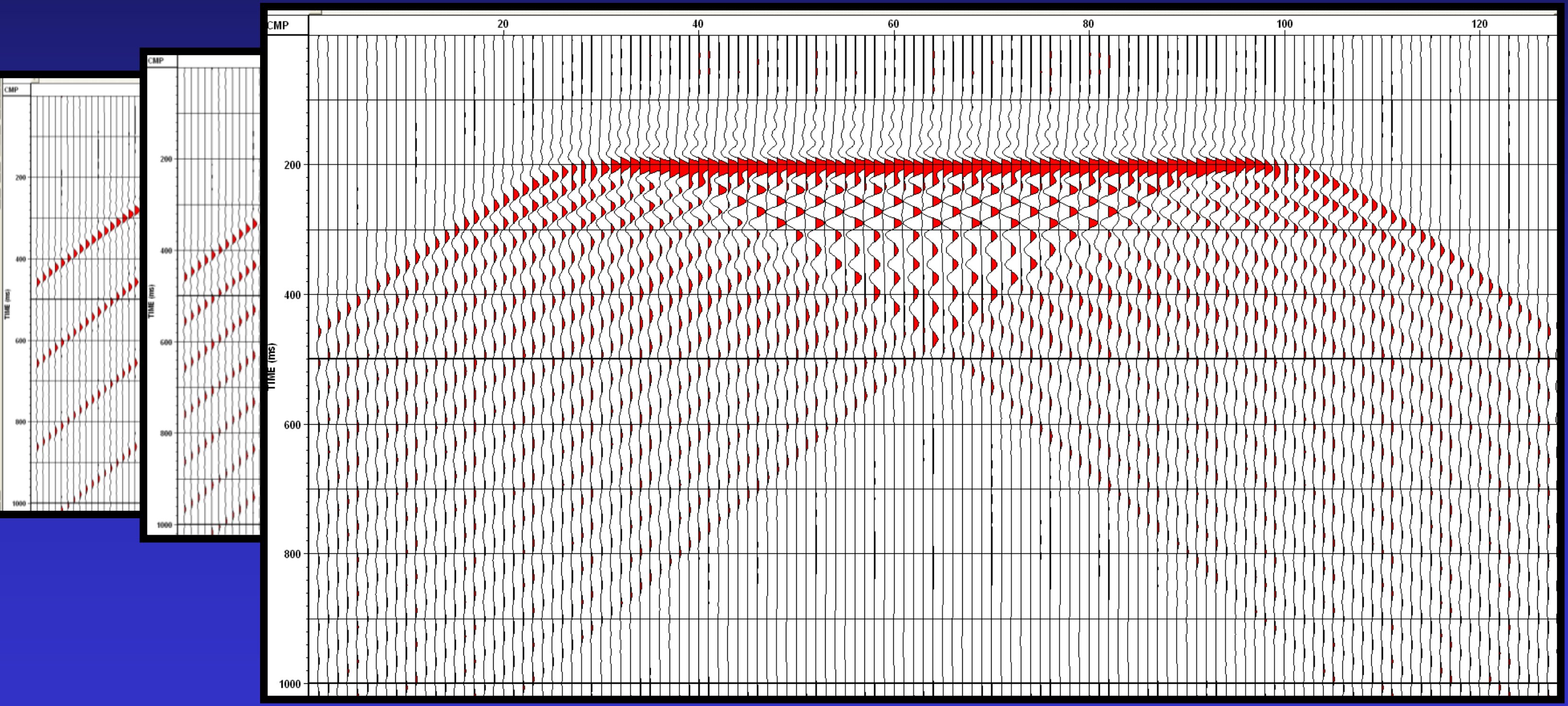

Modelling wavelet

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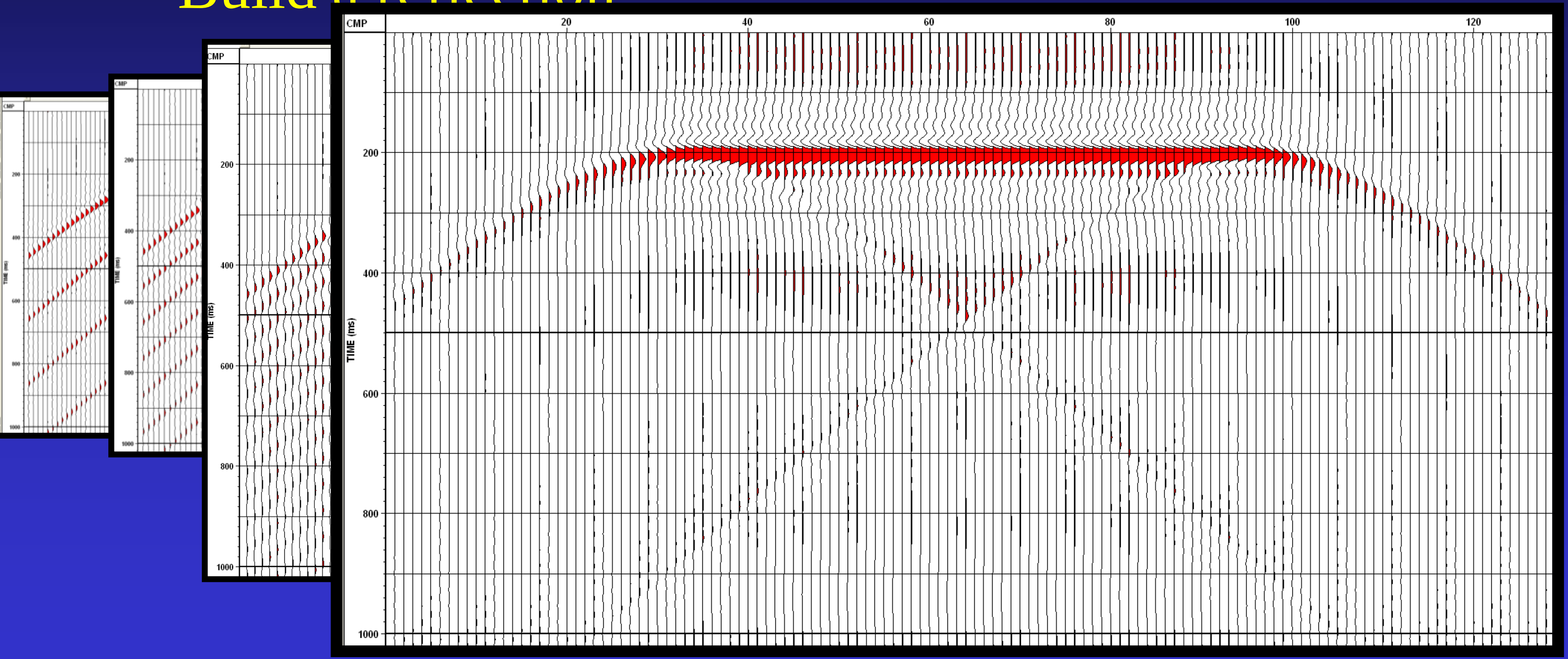

Modelling wavelet

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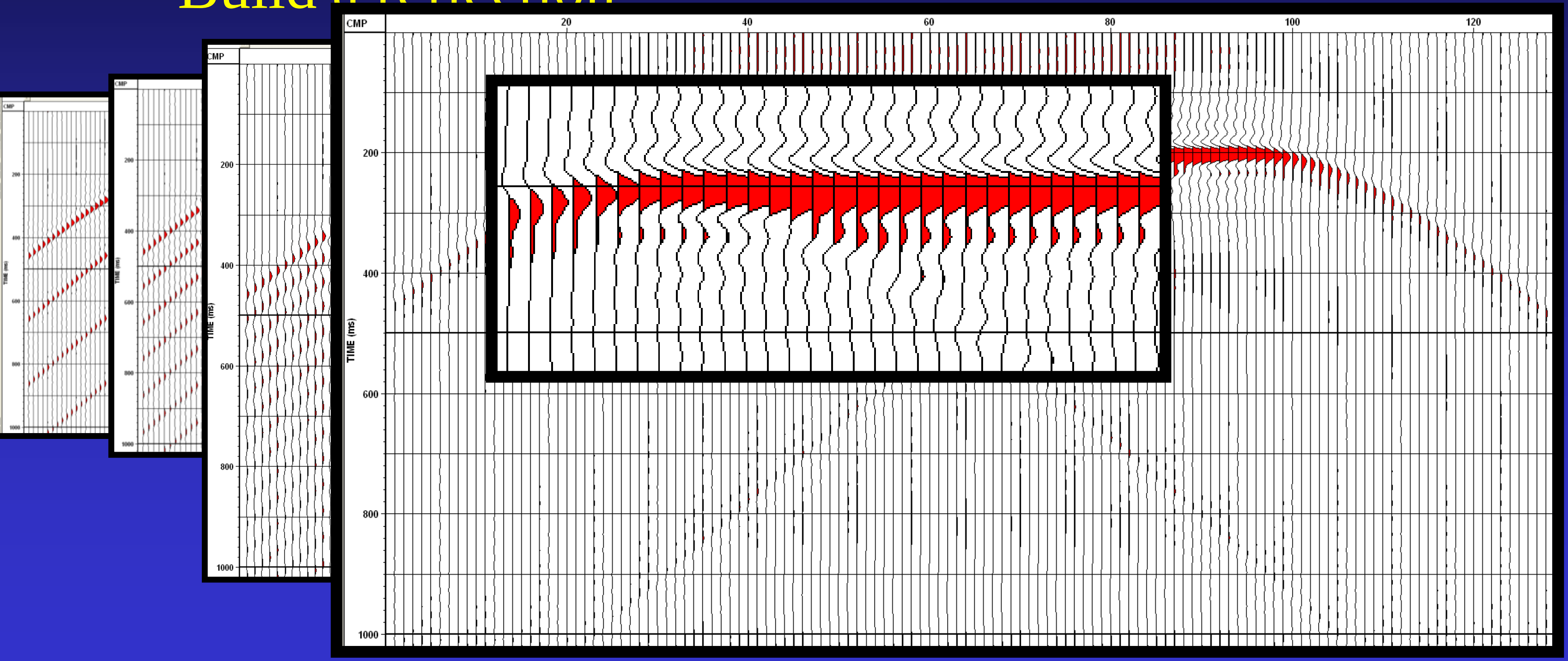
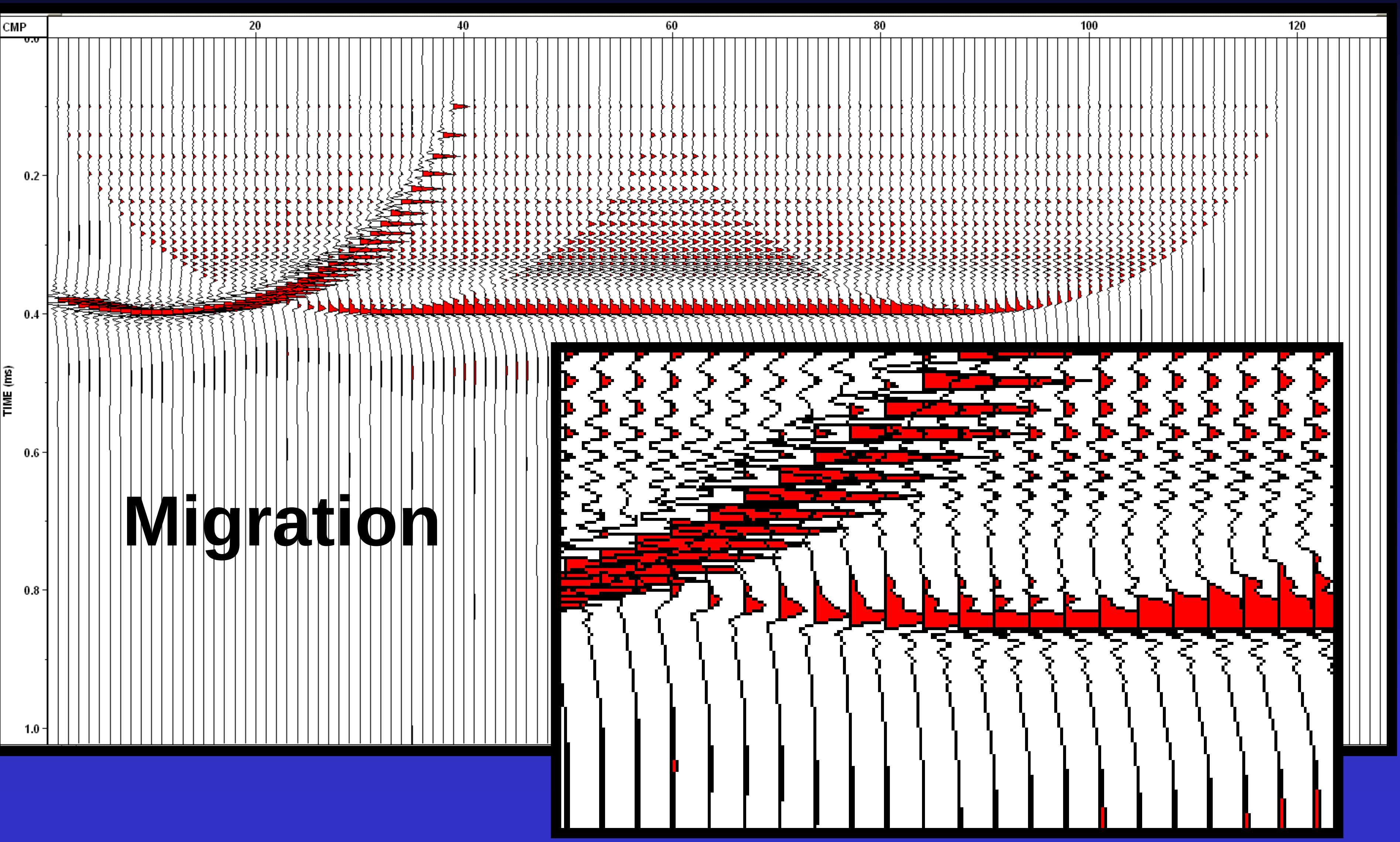
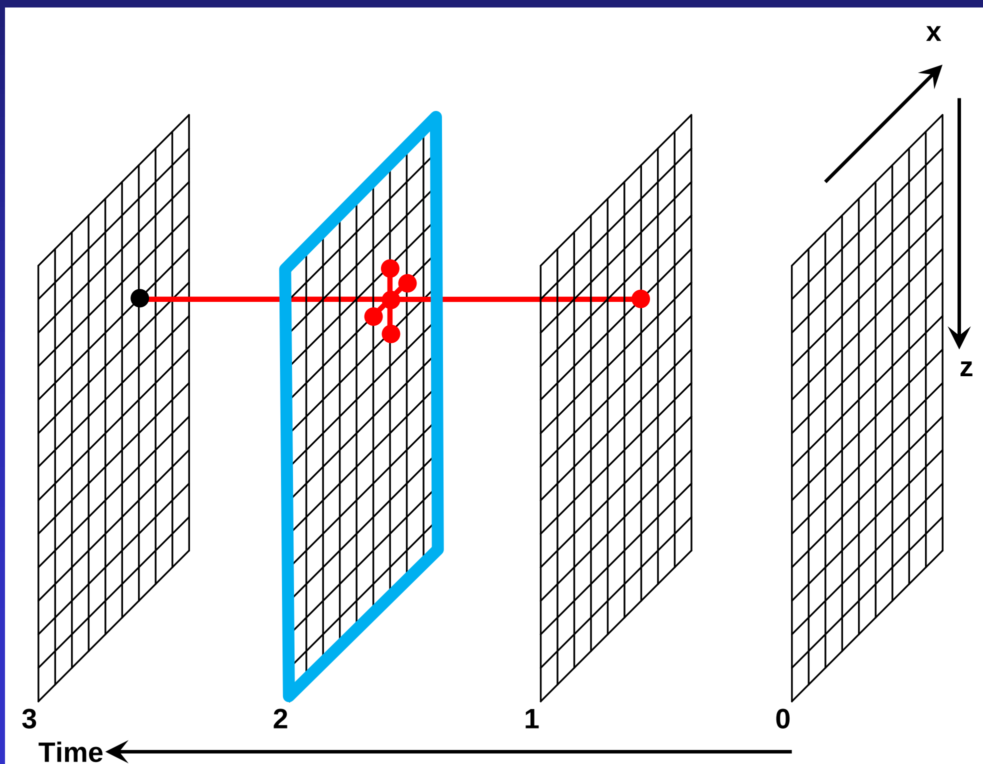


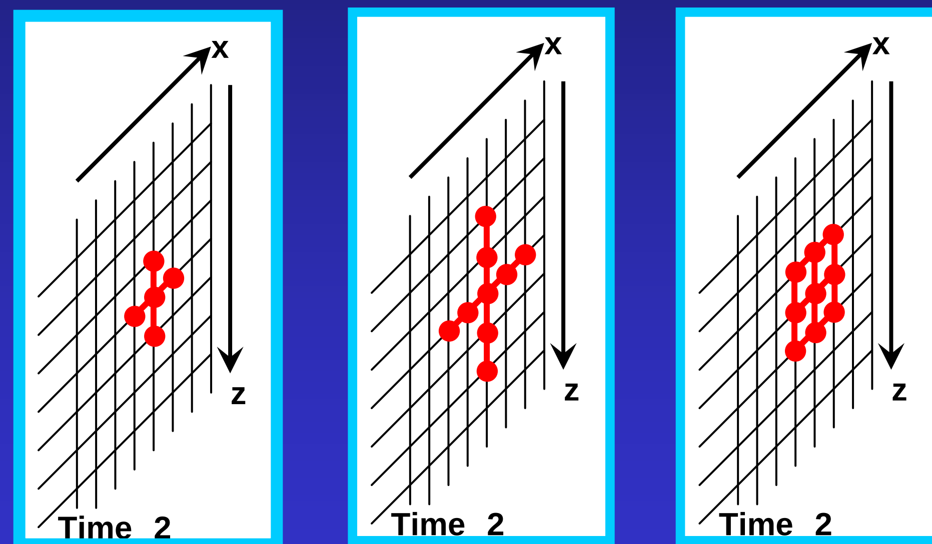
Migration wavelet:



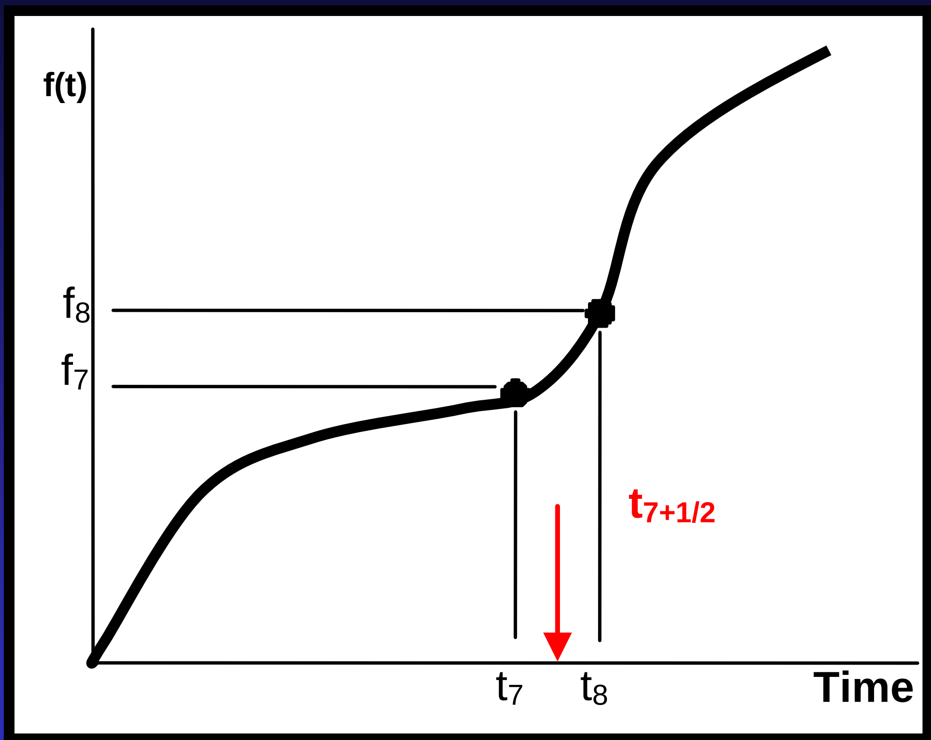
Solving the wave-equation:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2}$$

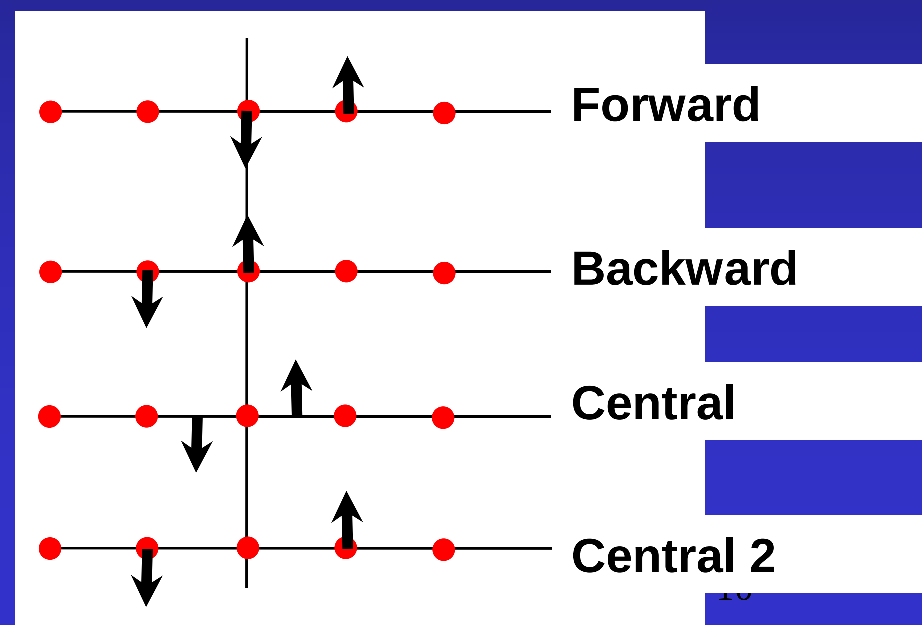




First derivative:



$$\left.\frac{d\,f(t)}{dt}\right|_{-t_7+t_8\to 0} \approx \frac{-f(t_7)+f(t_8)}{-t_7+t_8}$$



All have different properties

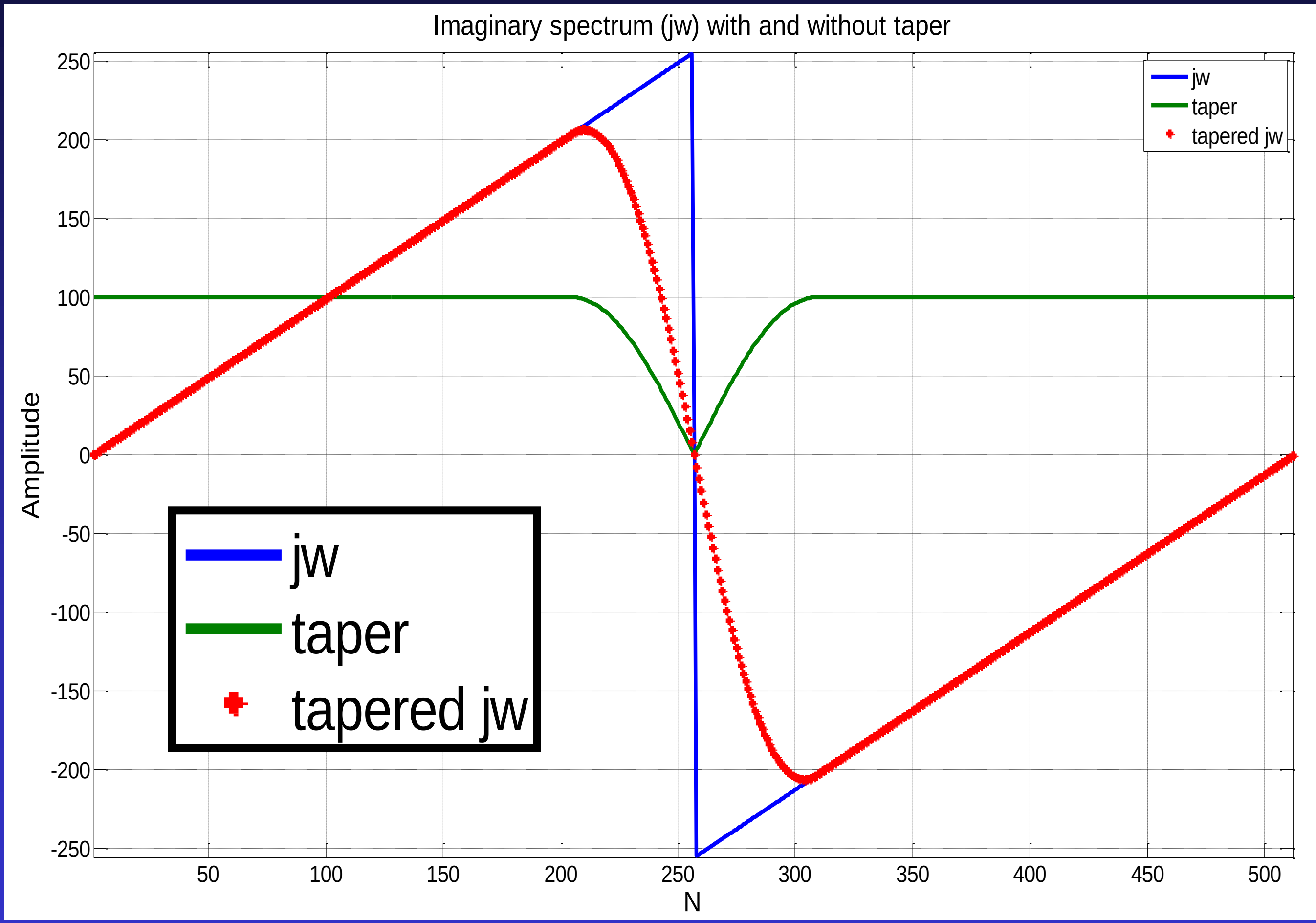
Correlation operators

First derivative: definition

$$\mathrm{F}\{f(t)\} \Leftrightarrow F(\omega)$$

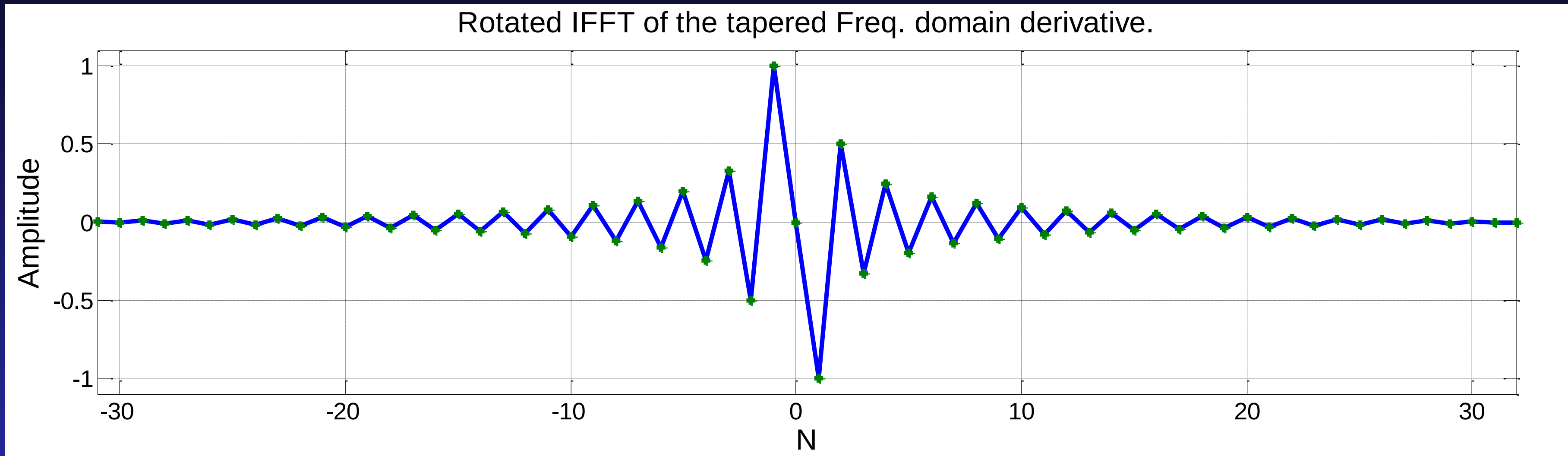
$$\mathrm{F}\left\{\frac{d\ f(t)}{dt}\right\} \Leftrightarrow j\omega F(\omega)$$

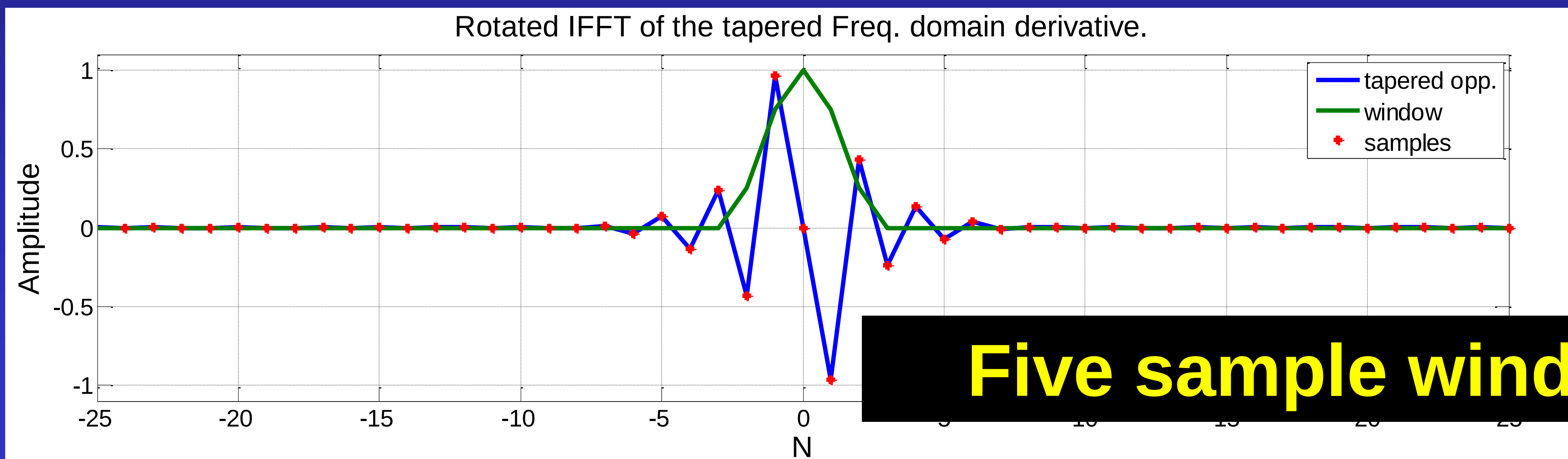
N = 512



Imaginary spectrum

First derivative in time:





Five sample window

First derivative: (algebra)

- Polynomial
- Taylor series

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + ... + q$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ... + \frac{h^n}{n!} f^n(x)$$

- Limit size of the polynomial
- Get exact derivatives
- Solve for the first derivative

First derivative

- Solve at $f(x-h)$, $f(x)$, and $f(x+h)$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

- Solve at $f(x)$, $f(x+h)$ and $f(x+2h)$

$$f'(x)_{\text{Left}} = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

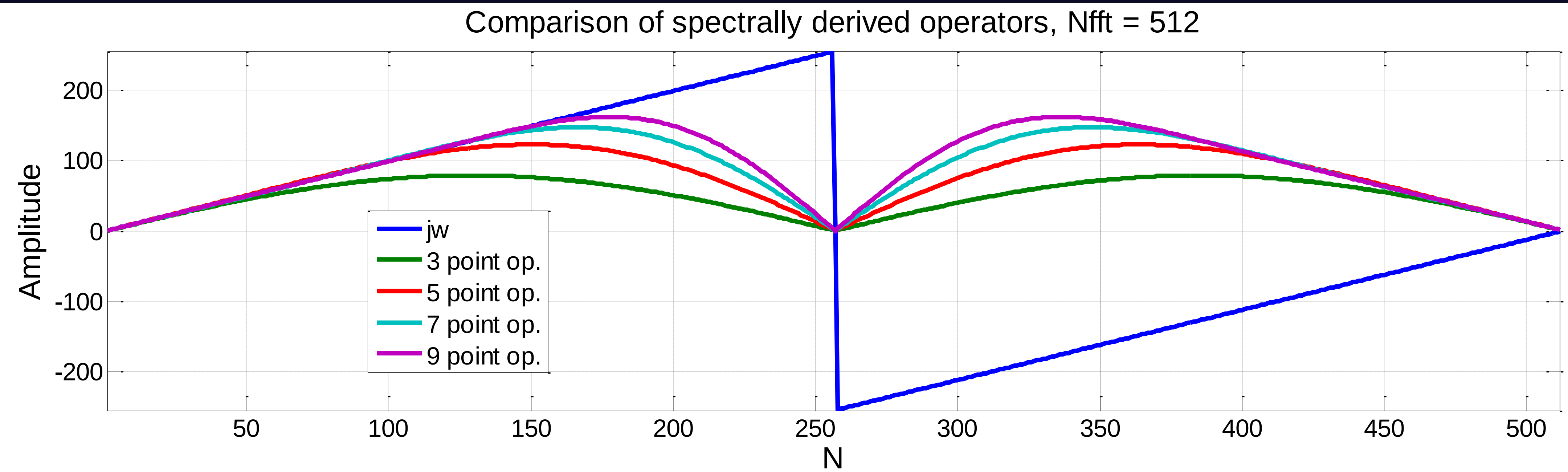
- Solve at $f(x-2h)$, $f(x-h)$, $f(x)$, $f(x+h)$ and $f(x+2h)$

$$f'(x) \approx \frac{+f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h}$$

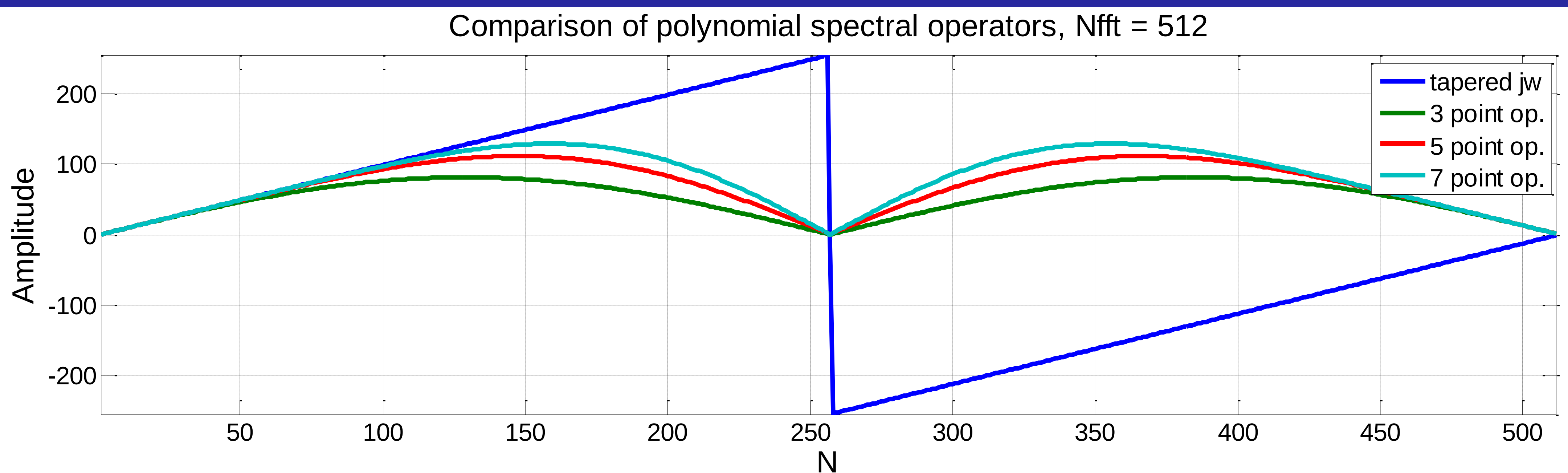
First de

Formula	Error term
$f'(x_i) \simeq \frac{f(x_{i-2})-4f(x_{i-1})+3f(x_i)}{2h}$	$\frac{1}{3}h^2 f^{(3)}$
$f'(x_i) \simeq \frac{f(x_{i+1})-f(x_{i-1})}{2h}$	$\frac{1}{6}h^2 f^{(3)}$
$f'(x_i) \simeq \frac{-3f(x_i)+4f(x_{i+1})-f(x_{i+2})}{2h}$	$\frac{1}{3}h^2 f^{(3)}$
$f'(x_i) \simeq \frac{3f(x_{i-4})-16f(x_{i-3})+36f(x_{i-2})-48f(x_{i-1})+25f(x_i)}{12h}$	$\frac{1}{5}h^4 f^{(5)}$
$f'(x_i) \simeq \frac{-f(x_{i-3})+6f(x_{i-2})-18f(x_{i-1})+10f(x_i)+3f(x_{i+1})}{12h}$	$\frac{1}{20}h^4 f^{(5)}$
$f'(x_i) \simeq \frac{f(x_{i-2})-8f(x_{i-1})+8f(x_{i+1})-f(x_{i+2})}{12h}$	$\frac{1}{30}h^4 f^{(5)}$
$f'(x_i) \simeq \frac{-3f(x_{i-1})-10f(x_i)+18f(x_{i+1})-6f(x_{i+2})+f(x_{i+3})}{12h}$	$\frac{1}{20}h^4 f^{(5)}$
$f'(x_i) \simeq \frac{-25f(x_i)+48f(x_{i+1})-36f(x_{i+2})+16f(x_{i+3})-3f(x_{i+4})}{12h}$	$\frac{1}{5}h^4 f^{(5)}$
$f'(x_i) \simeq \frac{10f(x_{i-6})-72f(x_{i-5})+225f(x_{i-4})-400f(x_{i-3})+450f(x_{i-2})-360f(x_{i-1})+147f(x_i)}{60h}$	$\frac{1}{7}h^6 f^{(7)}$
$f'(x_i) \simeq \frac{-2f(x_{i-5})+15f(x_{i-4})-50f(x_{i-3})+100f(x_{i-2})-150f(x_{i-1})+77f(x_i)+10f(x_{i+1})}{60h}$	$\frac{1}{42}h^6 f^{(7)}$
$f'(x_i) \simeq \frac{f(x_{i-4})-8f(x_{i-3})+30f(x_{i-2})-80f(x_{i-1})+35f(x_i)+24f(x_{i+1})-2f(x_{i+2})}{60h}$	$\frac{1}{105}h^6 f^{(7)}$
$f'(x_i) \simeq \frac{-f(x_{i-3})+9f(x_{i-2})-45f(x_{i-1})+45f(x_{i+1})-9f(x_{i+2})+f(x_{i+3})}{60h}$	$\frac{1}{140}h^6 f^{(7)}$
$f'(x_i) \simeq \frac{2f(x_{i-2})-24f(x_{i-1})-35f(x_i)+80f(x_{i+1})-30f(x_{i+2})+8f(x_{i+3})-f(x_{i+4})}{60h}$	$\frac{1}{105}h^6 f^{(7)}$
$f'(x_i) \simeq \frac{-10f(x_{i-1})-77f(x_i)+150f(x_{i+1})-100f(x_{i+2})+50f(x_{i+3})-15f(x_{i+4})+2f(x_{i+5})}{60h}$	$\frac{1}{42}h^6 f^{(7)}$
$f'(x_i) \simeq \frac{-147f(x_i)+360f(x_{i+1})-450f(x_{i+2})+400f(x_{i+3})-225f(x_{i+4})+72f(x_{i+5})-10f(x_{i+6})}{60h}$	$\frac{1}{7}h^6 f^{(7)}$

http://documents.wolfram.com/mathematica/Built-inFunctions/AdvancedDocumentation/DifferentialEquations/NDSolve/PartialDifferentialEquations/TheNumericalMethodOfLines/SpatialDerivativeApproximations.html

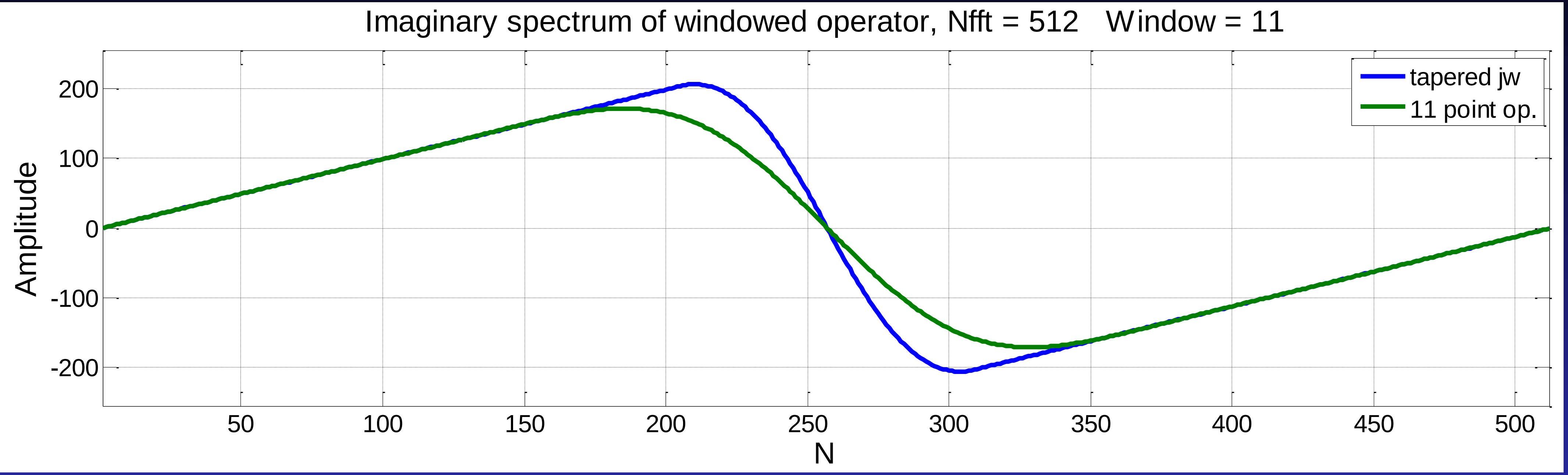


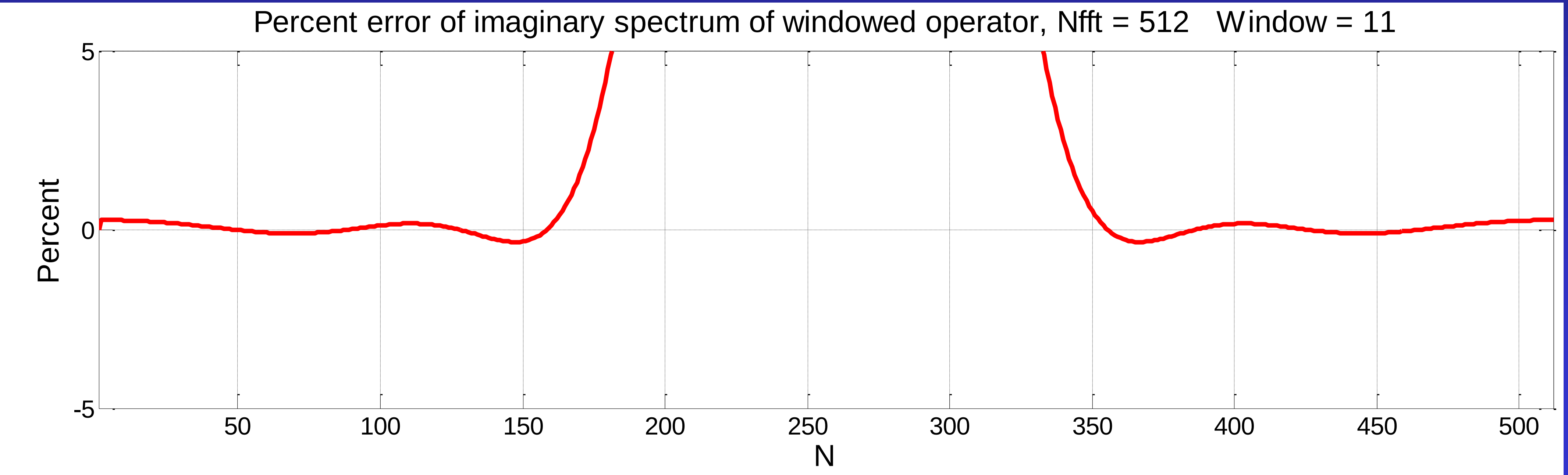
Comparison of spectrally derived operators, Nfft = 512
Amplitude
200
100
0
-100
-200
50
100
150
200
250
300
350
400
450
500
N
jw
3 point op.
5 point op.
7 point op.
9 point op.



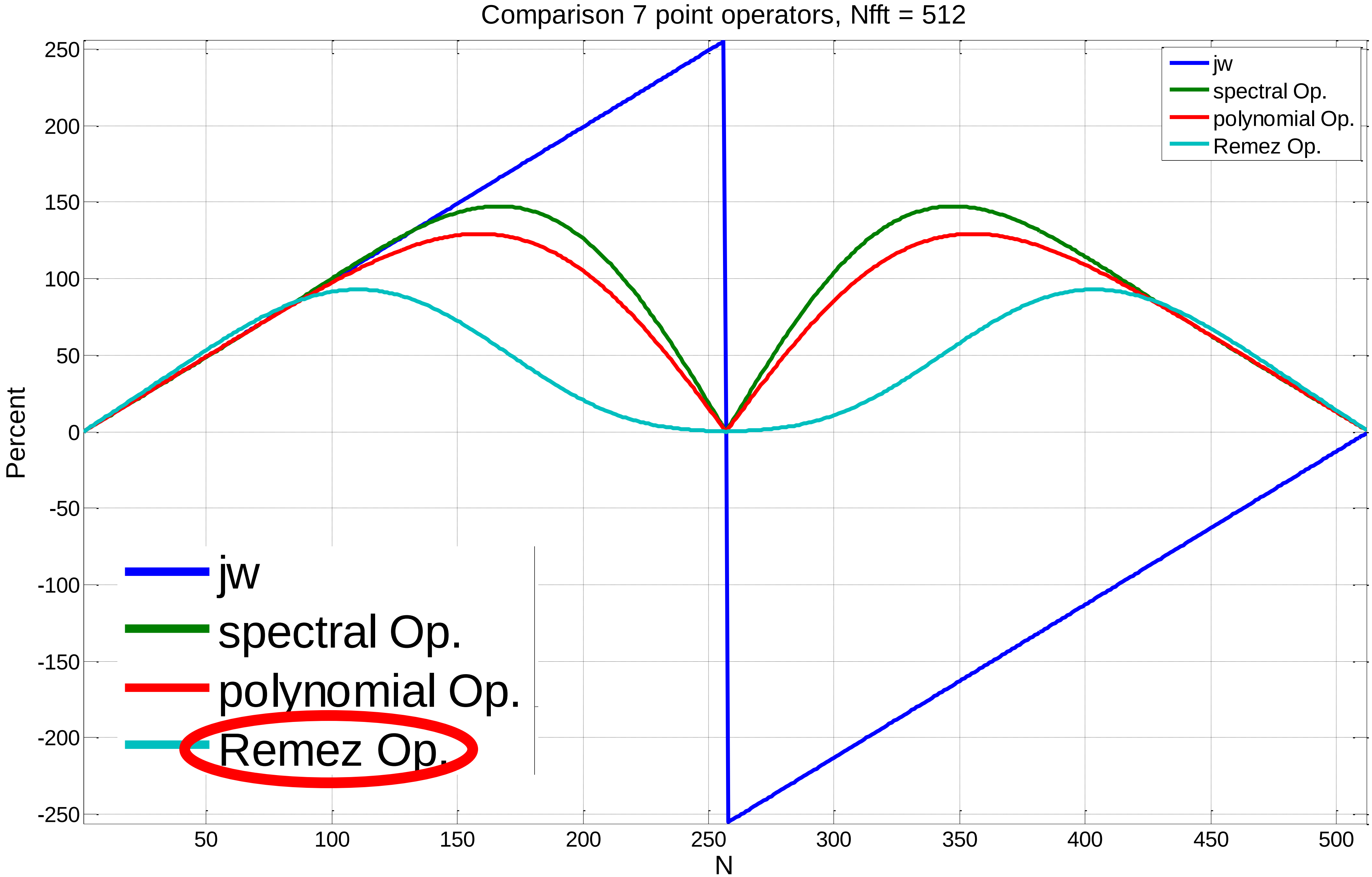
Comparison of polynomial spectral operators, Nfft = 512
Amplitude
200
100
0
-100
-200
50
100
150
200
250
300
350
400
450
500
N
tapered jw
3 point op.
5 point op.
7 point op.

First derivative





First derivative: seven point opers.



First derivative: other methods

- Remez algorithm
  - In MATLAB

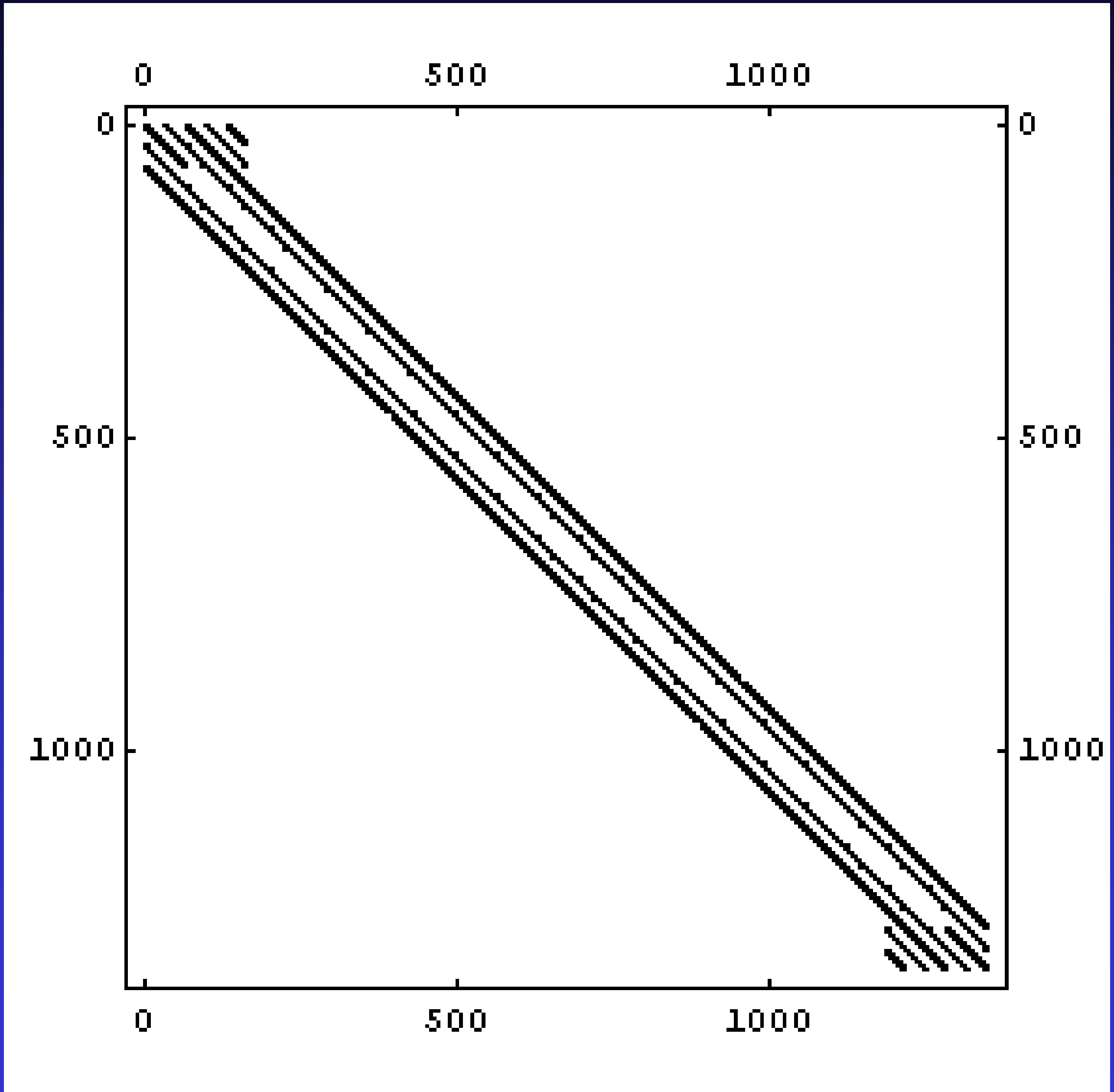
- Using Z transform
  - Finite difference
  - Z transform
  - Recursive response
  - Frequency response
  - Stability issues

$$h(z) = \frac{2(-1+z)}{(1+\gamma z)}$$

$$y_{n+1} = 2(x_{n+1} - x_n) - \gamma y_n$$

$$h(\omega) = \frac{2(e^{j\omega} - 1)}{(1+\gamma e^{j\omega})}$$

First derivative: matrix, five points



Second derivative: examples

Formula

$$f''(x_i) \simeq \frac{-f(x_{i-3})+4\,f(x_{i-2})-5\,f(x_{i-1})+2\,f(x_i)}{h^2}$$

$$f''(x_i) \simeq \frac{f(x_{i-1})-2\,f(x_i)+f(x_{i+1})}{h^2}$$

1 -2 1

$$f''(x_i) \simeq \frac{2\,f(x_i)-5\,f(x_{i+1})+4\,f(x_{i+2})-f(x_{i+3})}{h^2}$$

$$f''(x_i) \simeq \frac{-10\,f(x_{i-5})+61\,f(x_{i-4})-156\,f(x_{i-3})+214\,f(x_{i-2})-154\,f(x_{i-1})+45\,f(x_i)}{12\,h^2}$$

$$f''(x_i) \simeq \frac{f(x_{i-4})-6\,f(x_{i-3})+14\,f(x_{i-2})-4\,f(x_{i-1})-15\,f(x_i)+10\,f(x_{i+1})}{12\,h^2}$$

$$f''(x_i) \simeq \frac{-f(x_{i-2})+16\,f(x_{i-1})-30\,f(x_i)+16\,f(x_{i+1})-f(x_{i+2})}{12\,h^2}$$

$$f''(x_i) \simeq \frac{10\,f(x_{i-1})-15\,f(x_i)-4\,f(x_{i+1})+14\,f(x_{i+2})-6\,f(x_{i+3})+f(x_{i+4})}{12\,h^2}$$

$$f''(x_i) \simeq \frac{45\,f(x_i)-154\,f(x_{i+1})+214\,f(x_{i+2})-156\,f(x_{i+3})+61\,f(x_{i+4})-10\,f(x_{i+5})}{12\,h^2}$$

$$f''(x_i) \simeq \frac{-126\,f(x_{i-7})+1019\,f(x_{i-6})-3618\,f(x_{i-5})+7380\,f(x_{i-4})-9490\,f(x_{i-3})+7911\,f(x_{i-2})-4014\,f(x_{i-1})+938\,f(x_i)}{180\,h^2}$$

$$f''(x_i) \simeq \frac{11\,f(x_{i-6})-90\,f(x_{i-5})+324\,f(x_{i-4})-670\,f(x_{i-3})+855\,f(x_{i-2})-486\,f(x_{i-1})-70\,f(x_i)+126\,f(x_{i+1})}{180\,h^2}$$

$$f''(x_i) \simeq \frac{-2\,f(x_{i-5})+16\,f(x_{i-4})-54\,f(x_{i-3})+85\,f(x_{i-2})+130\,f(x_{i-1})-378\,f(x_i)+214\,f(x_{i+1})-11\,f(x_{i+2})}{180\,h^2}$$

$$f''(x_i) \simeq \frac{2\,f(x_{i-3})-27\,f(x_{i-2})+270\,f(x_{i-1})-490\,f(x_i)+270\,f(x_{i+1})-27\,f(x_{i+2})+2\,f(x_{i+3})}{180\,h^2}$$

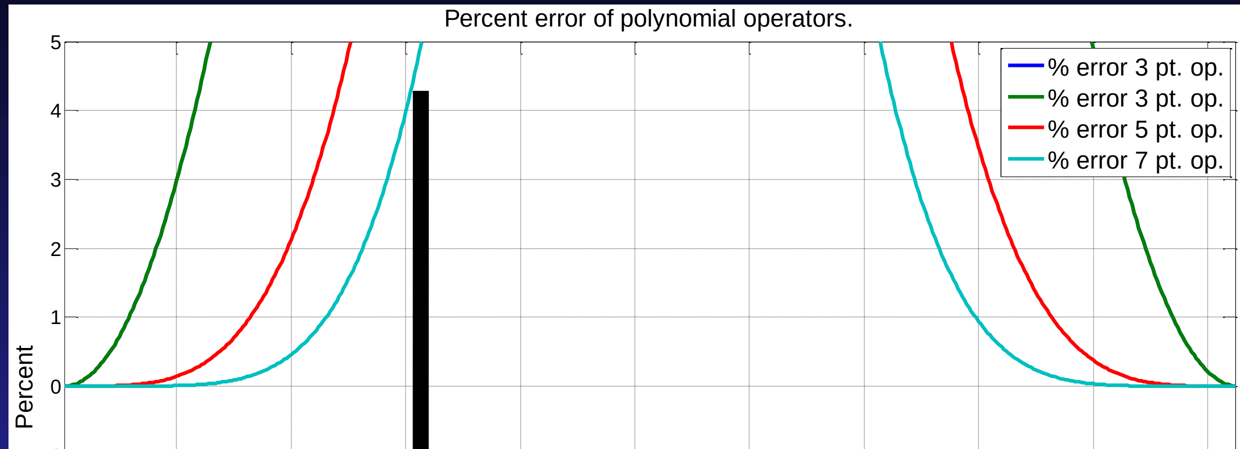
$$f''(x_i) \simeq \frac{-11\,f(x_{i-2})+214\,f(x_{i-1})-378\,f(x_i)+130\,f(x_{i+1})+85\,f(x_{i+2})-54\,f(x_{i+3})+16\,f(x_{i+4})-2\,f(x_{i+5})}{180\,h^2}$$

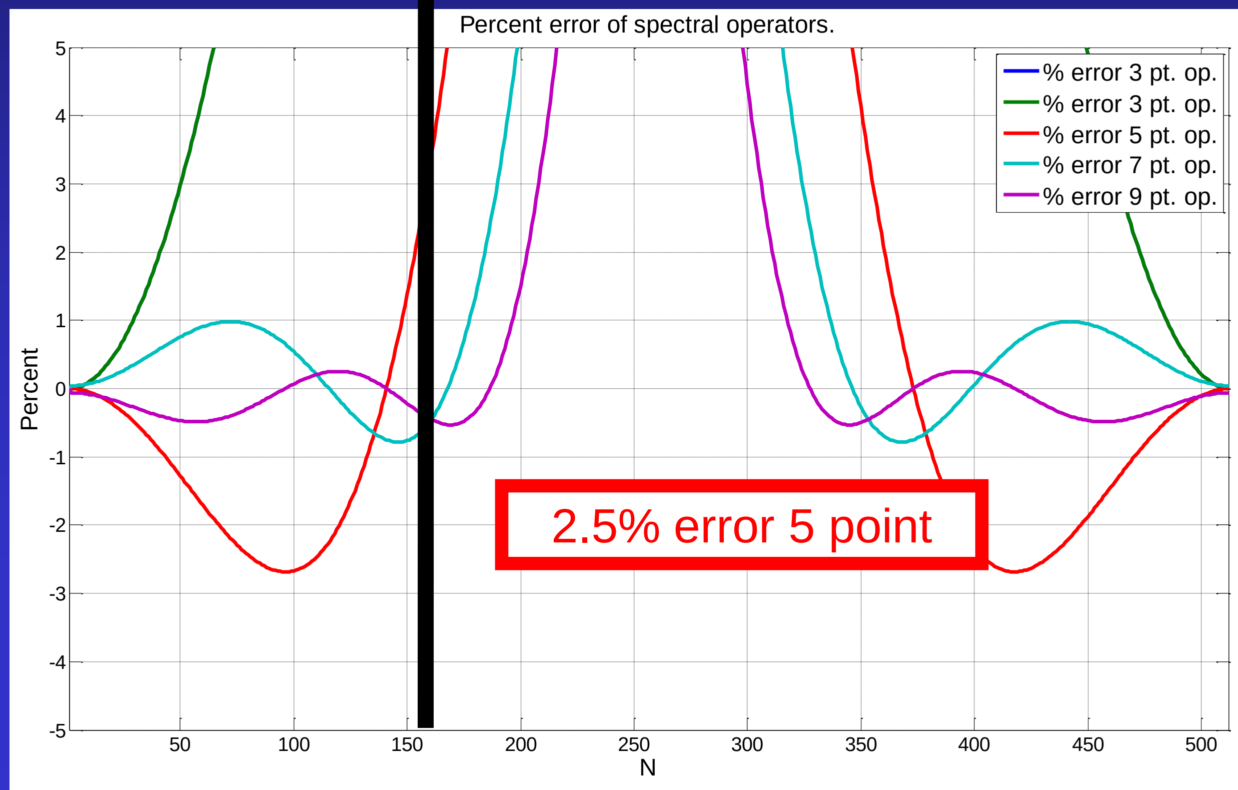
$$f''(x_i) \simeq \frac{126\,f(x_{i-1})-70\,f(x_i)-486\,f(x_{i+1})+855\,f(x_{i+2})-670\,f(x_{i+3})+324\,f(x_{i+4})-90\,f(x_{i+5})+11\,f(x_{i+6})}{180\,h^2}$$

$$f''(x_i) \simeq \frac{938\,f(x_i)-4014\,f(x_{i+1})+7911\,f(x_{i+2})-9490\,f(x_{i+3})+7380\,f(x_{i+4})-3618\,f(x_{i+5})+1019\,f(x_{i+6})-126\,f(x_{i+7})}{180\,h^2}$$

http://documents.wolfram.com/mathematica/Built-inFunctions/AdvancedDocumentation/DifferentialEquations/NDSolve/PartialDifferentialEquations/TheNumericalMethodOfLines/SpatialDerivativeApproximations.html

Second derivative: errors





Square-root derivative (rho filter)

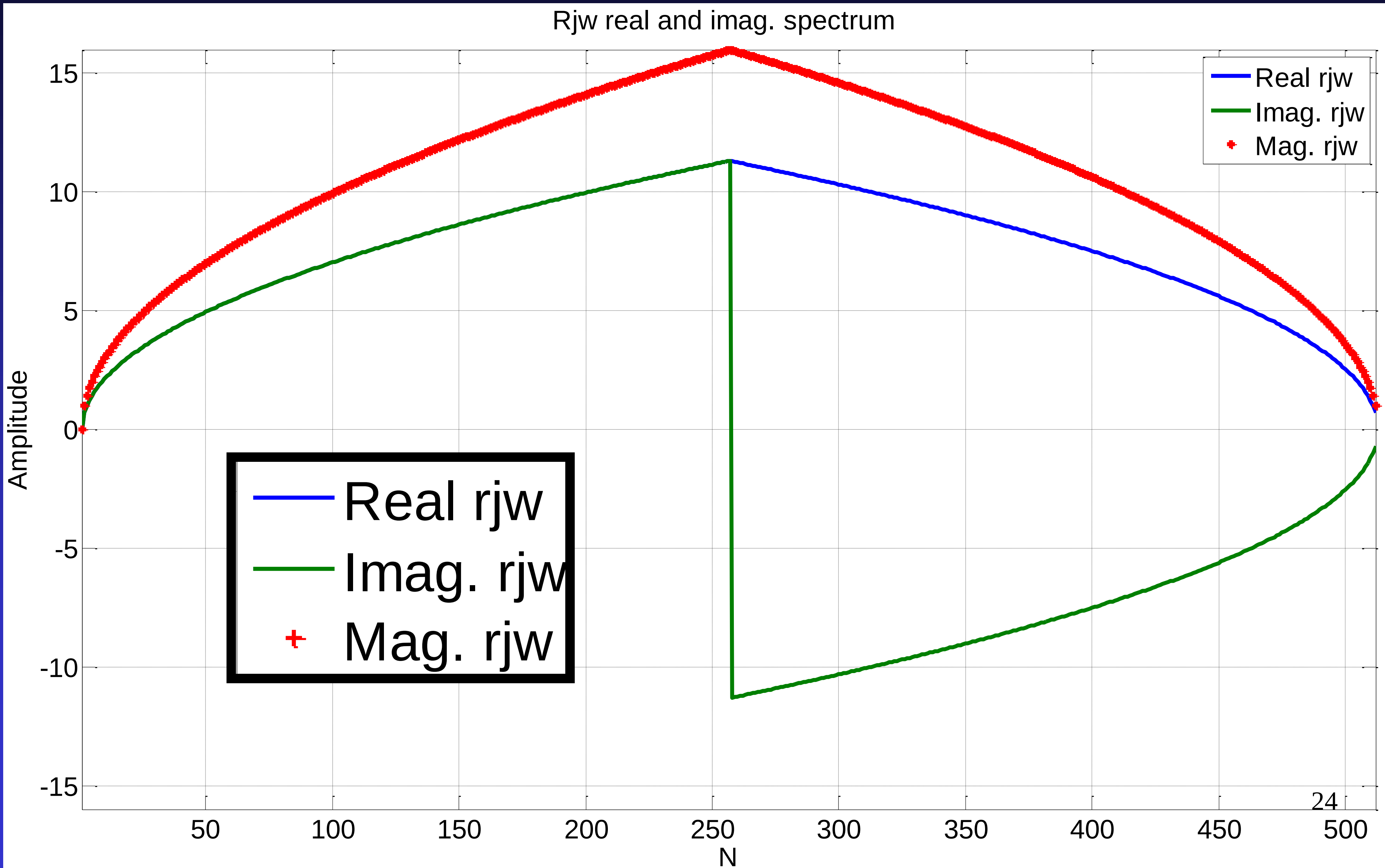
- What is it?

$$h(t) * h(t) = g(t) \qquad H(f) \times H(f) = H^2(f) = G(f)$$

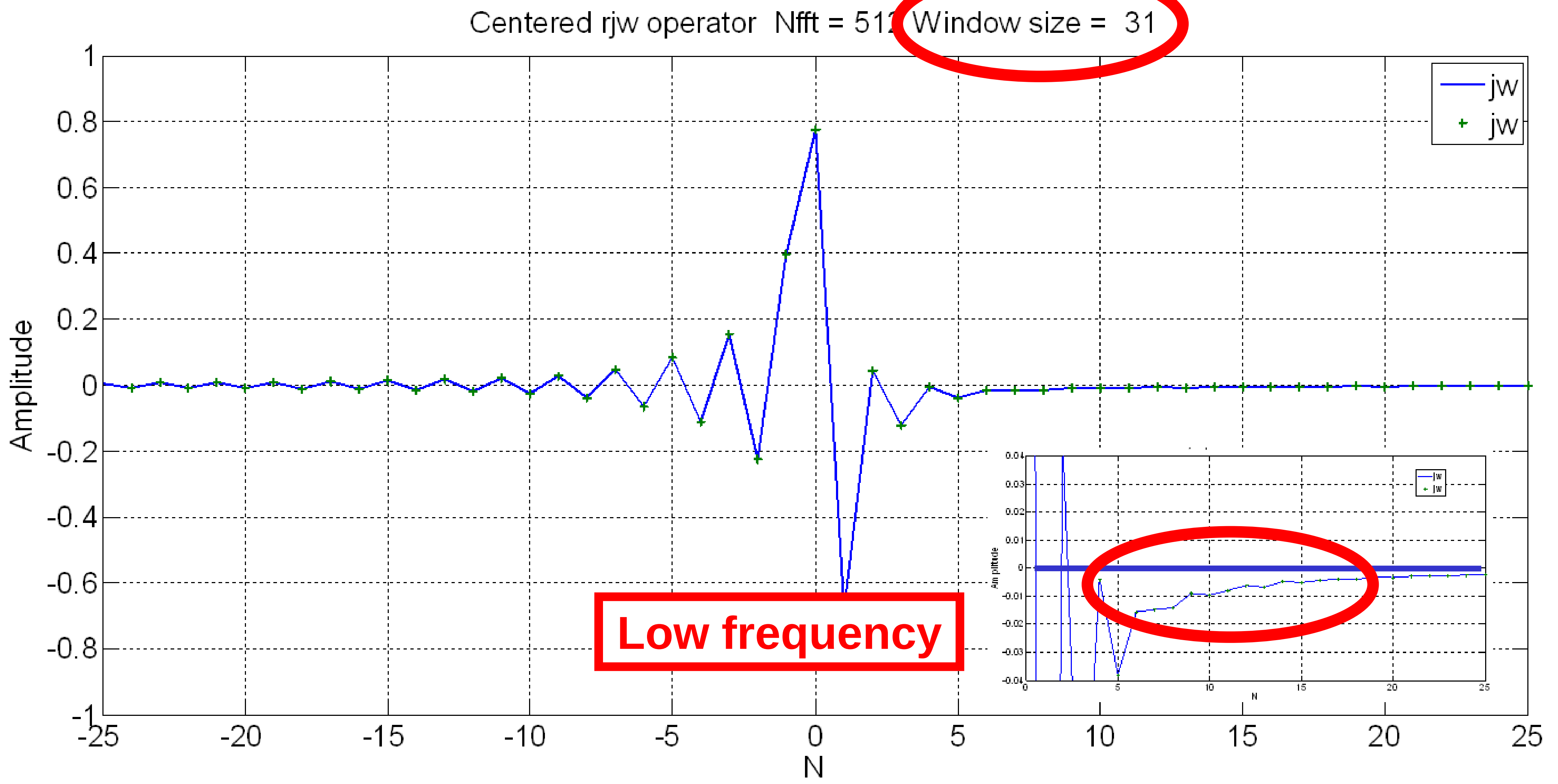
$$h(t) = \sqrt{g(t)} \qquad H(f) = \sqrt{G}(f)$$

- Apply twice to get the derivative

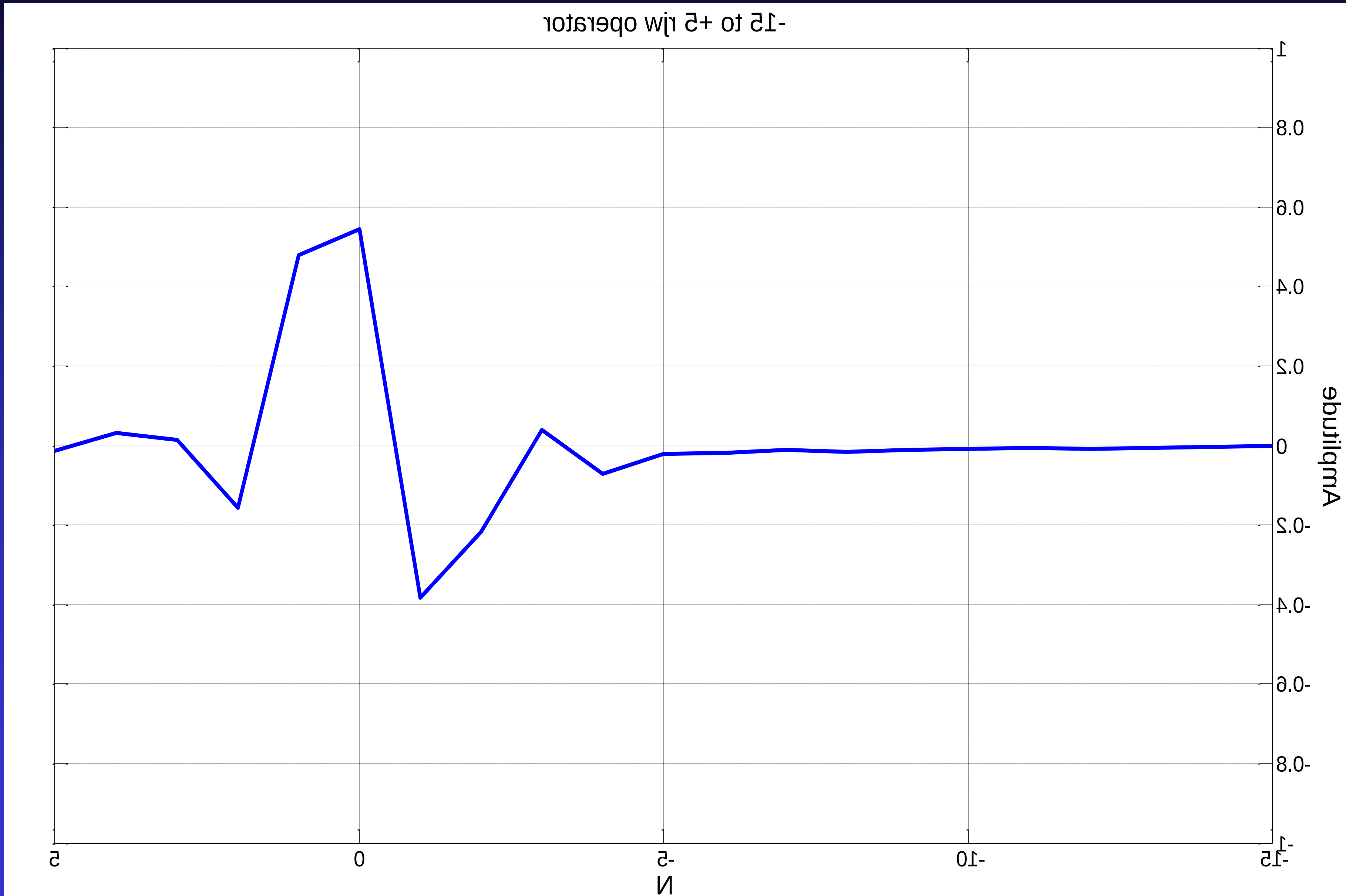
Square-root derivative: spectrum



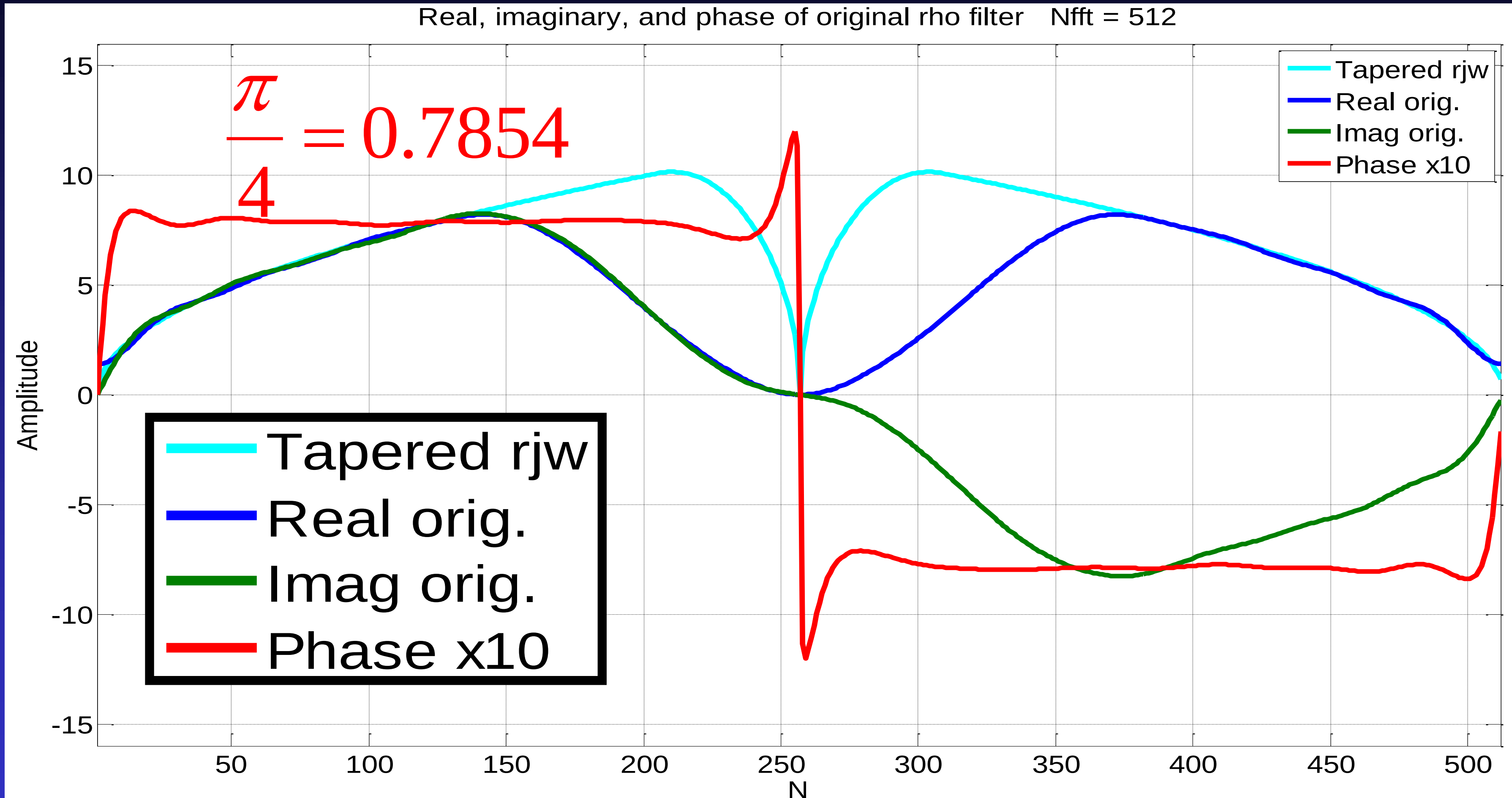
Square-root derivative: time

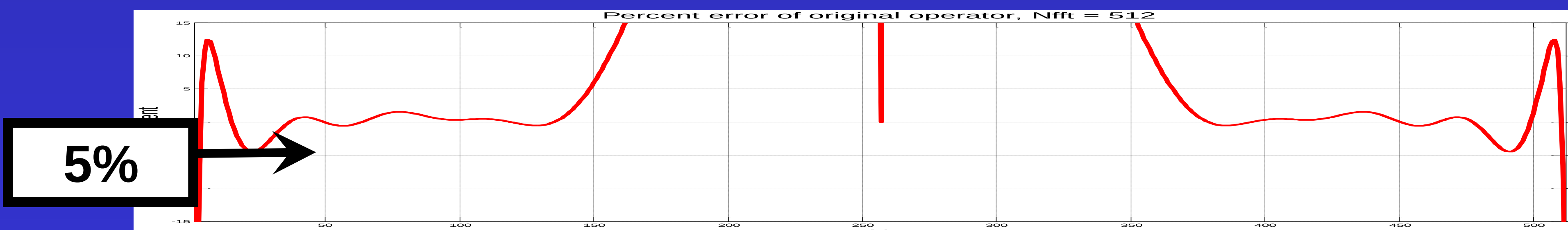


Square-root derivative   (-5 to +15)



Square-root derivative





Conclusions and comments

1. First, second, and square-root derivative

2. Polynomial derived
   - Better accuracy at low frequencies
   - Assume data is a low order polynomial

3. Spectrally derived
   - have greater bandwidth

4. Square-root
   - Still quite difficult
   - Experimenting with low freq. recursive filter

Thanks for your attention