Theory for direct absorptive AVA inversion

employing inverse series to characterize attenuating targets

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CREWES

Absorptive reflection coefficients

Medium 0:

$$\left[\nabla^2 + \frac{\omega^2}{c_0^2}\right] P_0(x, z, \omega) = 0$$

Medium 1:

$$\left[\nabla^2 + \frac{\omega^2}{c_1^2}\left(1 + \frac{F(\omega)}{Q_1}\right)^2\right] P_1(x, z, \omega) = 0$$

where

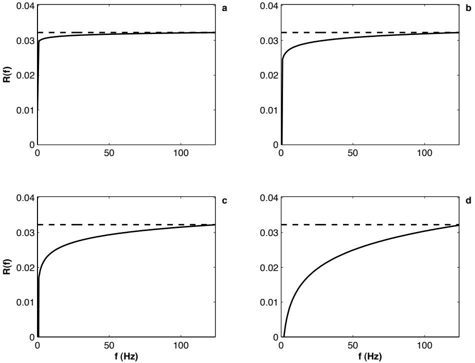
$$F(\omega) = \frac{i}{2} - \frac{1}{\pi}\log\left(\frac{\omega}{\omega_r}\right)$$

Absorptive reflection coefficients

At a plane boundary separating medium 0 and 1:

$$R = \frac{k_z - k_z'}{k_z + k_z'}$$

Q in the lower medium gives R some interesting characteristics. At normal incidence:



Absorptive reflection coefficients

We can express R in a number of ways. If ω is a parameter, R varies with θ as

$$R_\omega(\theta) = \frac{c_1 \cos\theta - c_0 \left[1 + Q_1^{-1} F(\omega)\right] \sqrt{1 - \frac{c_1^2}{c_0^2} \left[1 + Q_1^{-1} F(\omega)\right]^{-2} \sin^2\theta}}{c_1 \cos\theta + c_0 \left[1 + Q_1^{-1} F(\omega)\right] \sqrt{1 - \frac{c_1^2}{c_0^2} \left[1 + Q_1^{-1} F(\omega)\right]^{-2} \sin^2\theta}}$$

where

$$F(\omega) = \frac{i}{2} - \frac{1}{\pi} \log\left(\frac{\omega}{\omega_r}\right)$$

Absorptive reflection coefficients

...or, if k_z is a parameter, R varies with θ as

$$R_{kz}(\theta) = \frac{c_1 \cos\theta - c_0 \left[1 + Q_1^{-1} F_{kz}(\theta)\right] \sqrt{1 - \frac{c_1^2}{c_0^2} \left[1 + Q_1^{-1} F_{kz}(\theta)\right]^{-2} \sin^2\theta}}{c_1 \cos\theta + c_0 \left[1 + Q_1^{-1} F_{kz}(\theta)\right] \sqrt{1 - \frac{c_1^2}{c_0^2} \left[1 + Q_1^{-1} F_{kz}(\theta)\right]^{-2} \sin^2\theta}}$$

where

$$F_{kz}(\theta) = \frac{i}{2} - \frac{1}{\pi} \log\left(\frac{k_z c_0}{\omega_r \cos\theta}\right)$$

The forward problem and the inverse problem

Problem:

Given R at several values of these experimental variables, and the value of c in the incidence medium (medium 0), determine the values of c and Q in the target medium (medium 1).

Series expansions of R

Examples of quantities in R that are sometimes small, but not always, are the contrasts in wavespeed and Q from medium 0 to medium 1, and the angle of incidence θ.

Let us expand R about small

$$\alpha = 1 - \frac{c_0^2}{c_1^2}, \qquad \beta = \frac{1}{Q_1}, \qquad \text{and} \quad \sin^2\theta.$$

Series expansions of R

If ω is treated as a parameter, R expands as

$$
\begin{aligned}
&R_\omega(\theta)\\
&= \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{1}{8}\alpha^2 + \frac{1}{4}F^2(\omega)\beta^2\right) + \dots\right]\left(\sin^2\theta\right)^0\\
&+ \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{1}{4}\alpha^2 - \frac{1}{2}F(\omega)\alpha\beta + \frac{3}{4}F^2(\omega)\beta^2\right) + \dots\right]\left(\sin^2\theta\right)^1\\
&+ \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{5}{16}\alpha^2 - \frac{3}{4}F(\omega)\alpha\beta + F^2(\omega)\beta^2\right) + \dots\right]\left(\sin^2\theta\right)^2\\
&+ \dots \quad .
\end{aligned}
$$

Series expansions of R

Or, using

$$F_{kz}(\theta) = \frac{i}{2} - \frac{1}{\pi}\log\left(\frac{k_z c_0}{\omega_r \cos\theta}\right)$$

expand R as

$$\begin{aligned}
&R_{kz}(\theta) \\
&= \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F_{kz}(\theta)\beta\right) + \left(\frac{1}{8}\alpha^2 + \frac{1}{4}F_{kz}^2(\theta)\beta^2\right) + \ldots\right]\left(\sin^2\theta\right)^0 \\
&+ \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F_{kz}(\theta)\beta\right) + \left(\frac{1}{4}\alpha^2 - \frac{1}{2}F_{kz}(\theta)\alpha\beta + \frac{3}{4}F_{kz}^2(\theta)\beta^2\right) + \ldots\right]\left(\sin^2\theta\right)^1 \\
&+ \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F_{kz}(\theta)\beta\right) + \left(\frac{5}{16}\alpha^2 - \frac{3}{4}F_{kz}(\theta)\alpha\beta + F_{kz}^2(\theta)\beta^2\right) + \ldots\right]\left(\sin^2\theta\right)^2 \\
&+ \ldots \quad .
\end{aligned}$$

Direct AVF inversion for wavespeed and Q

Next, we form an inverse series for the unknown(s).

Begin with a simple example, to see how an inverse series solution operates (as well as convince ourselves it gives the right answer). Starting with $R_\omega(\theta)$, specify a normal incidence problem:

$$R_\omega(\theta)|_{\theta=0} = \left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{1}{8}\alpha^2 + \frac{1}{4}F^2(\omega)\beta^2\right) + \ldots$$

Direct AVF inversion for wavespeed and Q

Further, let us take the case in which only Q varies, i.e., $\alpha = 0$:

$$\begin{aligned} R_\omega &= -\frac{1}{2}F(\omega)\beta + \frac{1}{4}F^2(\omega)\beta^2 - \ldots \\ &= -\frac{\frac{1}{2}F(\omega)\beta}{1 + \frac{1}{2}F(\omega)\beta}. \end{aligned}$$

Direct AVF inversion for wavespeed and Q

...and solve for β; if the contrast is small, perhaps even linearize:

$$\begin{aligned}
\beta &= -\frac{2}{F(\omega)}\frac{R_\omega}{1+R_\omega} \\
&= -\frac{2}{F(\omega)}\left(R_\omega - R_\omega^2 + \ldots\right) \\
&\approx -\frac{2}{F(\omega)}R_\omega
\end{aligned}$$

Most important here is the series form of this *exact* solution.

Direct AVF inversion for wavespeed and Q

Try to recover this same answer by forming an inverse series for β:

$$\beta = \beta_1 + \beta_2 + \beta_3 + \ldots$$

where β_n is defined to be the component of β that is n'th order in the data (i.e., R_ω at whichever set of frequencies we employ).

Direct AVF inversion for wavespeed and Q

Substitute this into the forward problem:

$$\begin{aligned} R_\omega &= -\frac{1}{2}F(\omega)\beta + \frac{1}{4}F^2(\omega)\beta^2 - \ldots \\ &= -\frac{1}{2}F(\omega)[\beta_1 + \beta_2 + \ldots] + \frac{1}{4}F^2(\omega)[\beta_1 + \beta_2 + \ldots]^2 - \ldots \end{aligned}$$

Direct AVF inversion for wavespeed and Q

...and equate like orders:

$$\beta_1 = -\frac{2}{F(\omega)} R_\omega,$$

at first order,

$$\beta_2 = \frac{1}{2} F(\omega) \beta_1^2 = \frac{2}{F(\omega)} R_\omega^2,$$

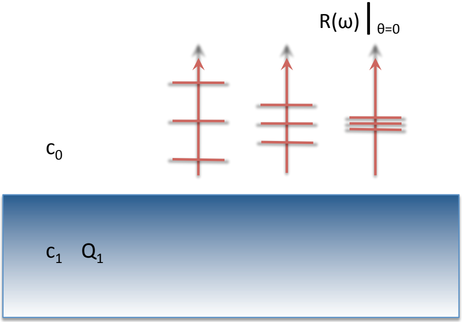
at second order, etc. Comparing to the expansion of the exact solution:

$$\beta = -\frac{2}{F(\omega)} \left(R_\omega - R_\omega^2 + ... \right)$$

...good news: we recover the correct answer.

Direct AVF inversion for wavespeed and Q

The "real" problem is to be able to determine simultaneous variations in c_1, Q_1. Given R at normal incidence, we may attempt this using its frequency dependence:



Direct AVF inversion for wavespeed and Q

Return to the normal incidence series for R_ω:

$$R_\omega(\theta)|_{\theta=0} = \left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{1}{8}\alpha^2 + \frac{1}{4}F^2(\omega)\beta^2\right) + ...$$

Form inverse series $\alpha = \alpha_1 + \alpha_2 + ...$ and $\beta = \beta_1 + \beta_2 + ...$, and substitute:

$$R_\omega = \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 - \frac{1}{2}F(\omega)\beta_1 - \frac{1}{2}F(\omega)\beta_2 + \frac{1}{8}\alpha_1^2 + \frac{1}{4}F^2(\omega)\beta_1^2 + ...,$$

Direct AVF inversion for wavespeed and Q

Equating like orders:

$$\begin{aligned} R_\omega &= \frac{1}{4}\alpha_1 - \frac{1}{2}F(\omega)\beta_1, \\ 0 &= \frac{1}{4}\alpha_2 - \frac{1}{2}F(\omega)\beta_2 + \frac{1}{8}\alpha_1^2 + \frac{1}{4}F^2(\omega)\beta_1^2, \\ &\ldots \end{aligned}$$

etc. Now we must cope with the fact that variations in either α or β may have affected R. Notice, however, that if we take R_ω at two different frequencies and subtract, we isolate the influence (to first order) of β_1.

Direct AVF inversion for wavespeed and Q

In fact,

$$\alpha_1 = 4\frac{R_{\omega_1}F(\omega_2) - R_{\omega_2}F(\omega_1)}{F(\omega_2) - F(\omega_1)},$$
$$\beta_1 = 2\frac{R_{\omega_1} - R_{\omega_2}}{F(\omega_2) - F(\omega_1)},$$

We may be willing to make the equivalent of the inverse Born approximation $\alpha \approx \alpha_1$, $\beta \approx \beta_1$ at this stage.

Direct AVF inversion for wavespeed and Q

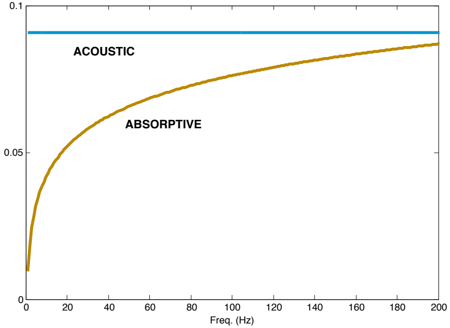
But we may also continue, if the contrasts causing R are thought to be large. Solving for α_2 and β_2 in terms of α_1 and β_1, we introduce a correction to the linear inversion, obtaining a direct, non-linear formula for c_1 and Q_1.

$$\alpha \approx \alpha_1 + \alpha_2 = \alpha_1 - \frac{1}{2}\alpha_1^2 - \beta_1^2 \left(\frac{F^2(\omega_1)F(\omega_2) - F^2(\omega_2)F(\omega_1)}{F(\omega_2) - F(\omega_1)} \right),$$

$$\beta \approx \beta_1 + \beta_2 = \beta_1 + \frac{1}{2}[F(\omega_2) + F(\omega_1)]\beta_1^2.$$

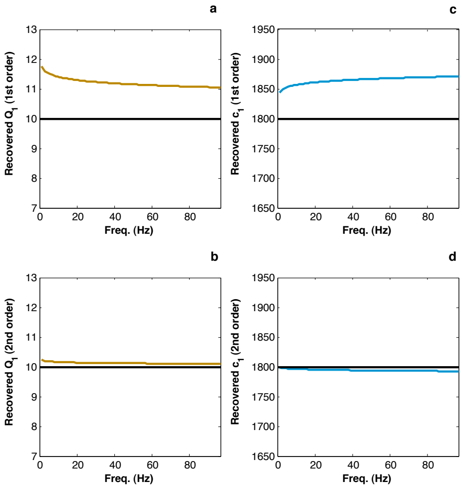
Direct AVF inversion for wavespeed and Q

Numerically how do the linear and non-linear formulas behave? If we choose c_0 =1500m/s, c_1 =1800m/s, Q_1 =10, the real part of R looks like:



Direct AVF inversion for wavespeed and Q

Next, fix ω_1 at 1Hz, vary ω_2 from 2-100Hz, and invert using each (ω_1, ω_2) pair. The linear inverse and the direct non-linear correction then look like:



Direct AVA inversion for wavespeed and Q: a quick look

Forming again inverse series for α and β, to first order we obtain

$$\beta \approx \beta_1 = 2\frac{R_{kz}(\theta_1)\cos^2\theta_1 - R_{kz}(\theta_2)\cos^2\theta_2}{F_{kz}(\theta_2) - F_{kz}(\theta_1)},$$

$$\alpha \approx \alpha_1 = 4\frac{R_{kz}(\theta_1)\cos^2\theta_1 F_{kz}(\theta_2) - R_{kz}(\theta_2)\cos^2\theta_2 F_{kz}(\theta_1)}{F_{kz}(\theta_2) - F_{kz}(\theta_1)}.$$

Direct AVA inversion for wavespeed and Q: a quick look

...and to second order

$$\beta \approx \beta_1 + \beta_2 = 2\frac{\tilde{R}_{kz}(\theta_1)\cos^2\theta_1 - \tilde{R}_{kz}(\theta_2)\cos^2\theta_2}{F_{kz}(\theta_2) - F_{kz}(\theta_1)}$$

$$\alpha \approx \alpha_1 + \alpha_2 = 4\frac{\tilde{R}_{kz}(\theta_1)F_{kz}(\theta_2)\cos^2\theta_1 - \tilde{R}_{kz}(\theta_2)F_{kz}(\theta_1)\cos^2\theta_2}{F_{kz}(\theta_2) - F_{kz}(\theta_1)},$$

where

$$\tilde{R}_{kz}(\theta) = R_{kz}(\theta) - \frac{1}{\cos^2\theta}\left(\frac{1}{4}\alpha_1^2 + \frac{1}{2}F_{kz}^2(\theta)\beta_1^2\right) - \left(\frac{1}{2}\alpha_1 + F_{kz}(\theta)\beta_1\right)^2\sin^2\theta.$$

Comments and next steps

- As seismic measurements (in particular amplitudes) become more precisely measured, direct, linear/non-linear formulas may be derived for direct determination of target absorptive properties
- These "inverse series" results are a special, simplified form of inverse scattering
- Next steps: anelastic theory, field-estimated R from absorptive targets

Thanks

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