UNIVERSITY OF CALGARY

CREWES

Feasibility of solving least-squares Gazdag migration using method of multigrid.

2010 CREWES
Sponsor Meeting

Abdolnaser Yousefzadeh, John C. Bancroft

Outline

I. Review on Least Squares Kirchhoff Migration/Inversion (LSM)

II. Least Squares Gazdag Migration/Inversion (LSM)

III. Multigrid methods

IV. Review on feasibility of Solving Kirchhoff LSM by Multigrid Methods

V. Feasibility of Solving Gazdag LSM by Multigrid Methods

VI. Summary

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VI. Summary

Kirchhoff Migration

✓ Kirchhoff is an industry standard.

✓ Kirchhoff works with

  - Incomplete data,
  - Irregular data

○ Incomplete/spars data: artifacts.

Replacing Migration with LSM:

- Remove migration artifacts/Attenuate acquisition footprints
- Provide high resolution images
- Compute images that can reproduce data

o Disadvantages:

  - Strongly depends on the velocity accuracy
  - Cost

Nemeth et. al. (1999 ,2000) and Duquet et. al. (2000).

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Gazdag Migration

- ✓ Gazdag migration works in the frequency domain.
- ✓ Fast.
- o Needs a regularly sampled data.
- o Not good for steeper dips.

Replacing Migration with LSM:

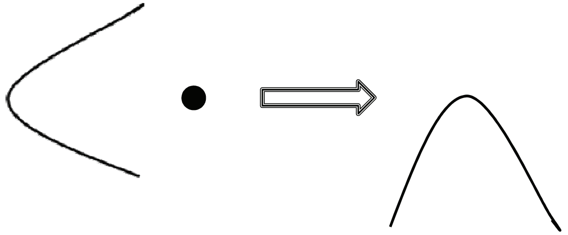
- Remove migration artifacts/Attenuate acquisition footprints.
- Better recovery of dipping events.
- Can work with incomplete data (by introducing a weight function).

- Disadvantages:
  - Cost

LSM, how?

- Modelling:

$$Gm = d$$



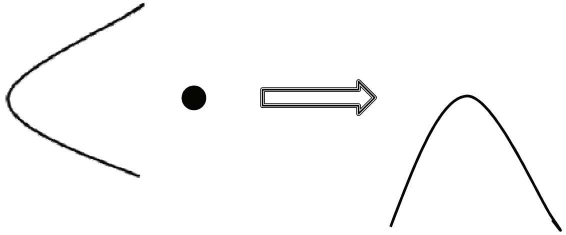
d : Real data,

m : Reflectivity,

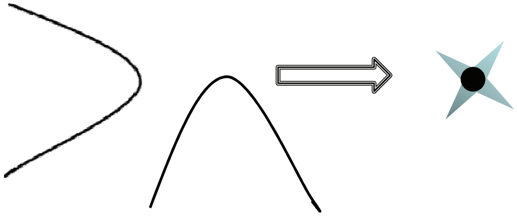
G : Kirchhoff/ Gazdag forward operator.

LSM, how?

- Modelling: $\mathbf{G}\mathbf{m} = \mathbf{d}$



- Migration: $\mathbf{G}^{T}\mathbf{d} = \hat{\mathbf{m}}$



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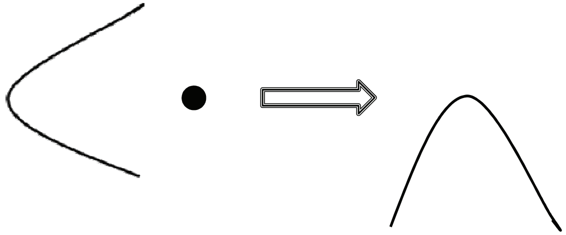
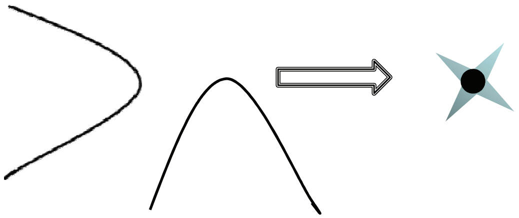
LSM, how?

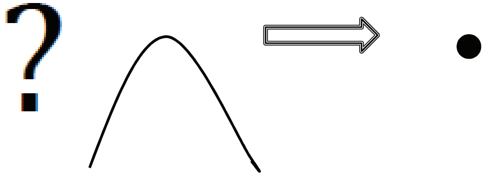
- Modelling: $$Gm = d$$



- Migration: $$G^{T} d = \hat{m}$$



- Inversion: $$G^{-1} d = m$$



d : Real data,

m : Reflectivity,

G : Kirchhoff/
Gazdag
forward
operator.

LSM, how?

$$Gm = d$$

$$G^T G\, m = G^T d$$

$$m = (G^T G)^{-1} G^T d$$

LSM, how?

$$\boldsymbol{Gm} = \boldsymbol{d}$$

$$\boldsymbol{G}^T \boldsymbol{G}\, \boldsymbol{m} = \boldsymbol{G}^T \boldsymbol{d}$$

$$\boldsymbol{m} = (\boldsymbol{G}^T \boldsymbol{G})^{-1} \boldsymbol{G}^T \boldsymbol{d}$$

- Minimizing a general cost function:

$$J(\boldsymbol{m}) = ||\boldsymbol{Gm} - \boldsymbol{d}||^2 + \mu^2\, \mathcal{R}(\boldsymbol{m})$$

$\mathcal{R}$: Regularization term,

μ : Trade-off parameter

LSM, how?

$$\boldsymbol{Gm} = \boldsymbol{d}$$

$$\boldsymbol{G}^T \boldsymbol{G}\, \boldsymbol{m} = \boldsymbol{G}^T \boldsymbol{d}$$

$$\boldsymbol{m} = (\boldsymbol{G}^T \boldsymbol{G})^{-1} \boldsymbol{G}^T \boldsymbol{d}$$

- Minimizing a general cost function:

$$J(\boldsymbol{m}) = ||\boldsymbol{Gm} - \boldsymbol{d}||^2 + \mu^2\, \mathcal{R}(\boldsymbol{m})$$

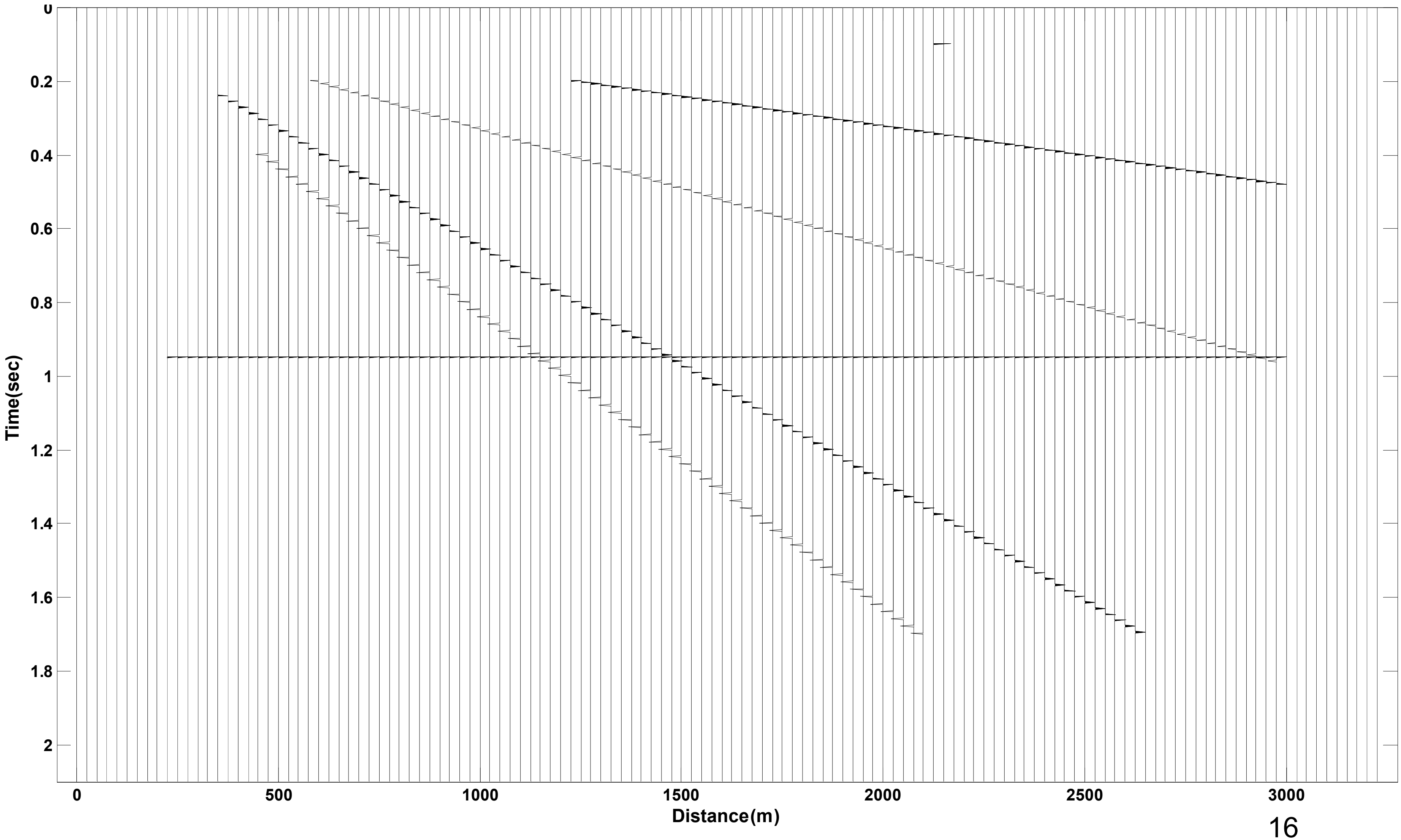
$$\mathcal{R}(\boldsymbol{m}) = ||\ \boldsymbol{m}\ ||_2^2$$

Damped: $\boldsymbol{m}_{DLS} = (\boldsymbol{G}^T \boldsymbol{G} + \mu^2 \boldsymbol{I})^{-1} \boldsymbol{G}^T \boldsymbol{d}$

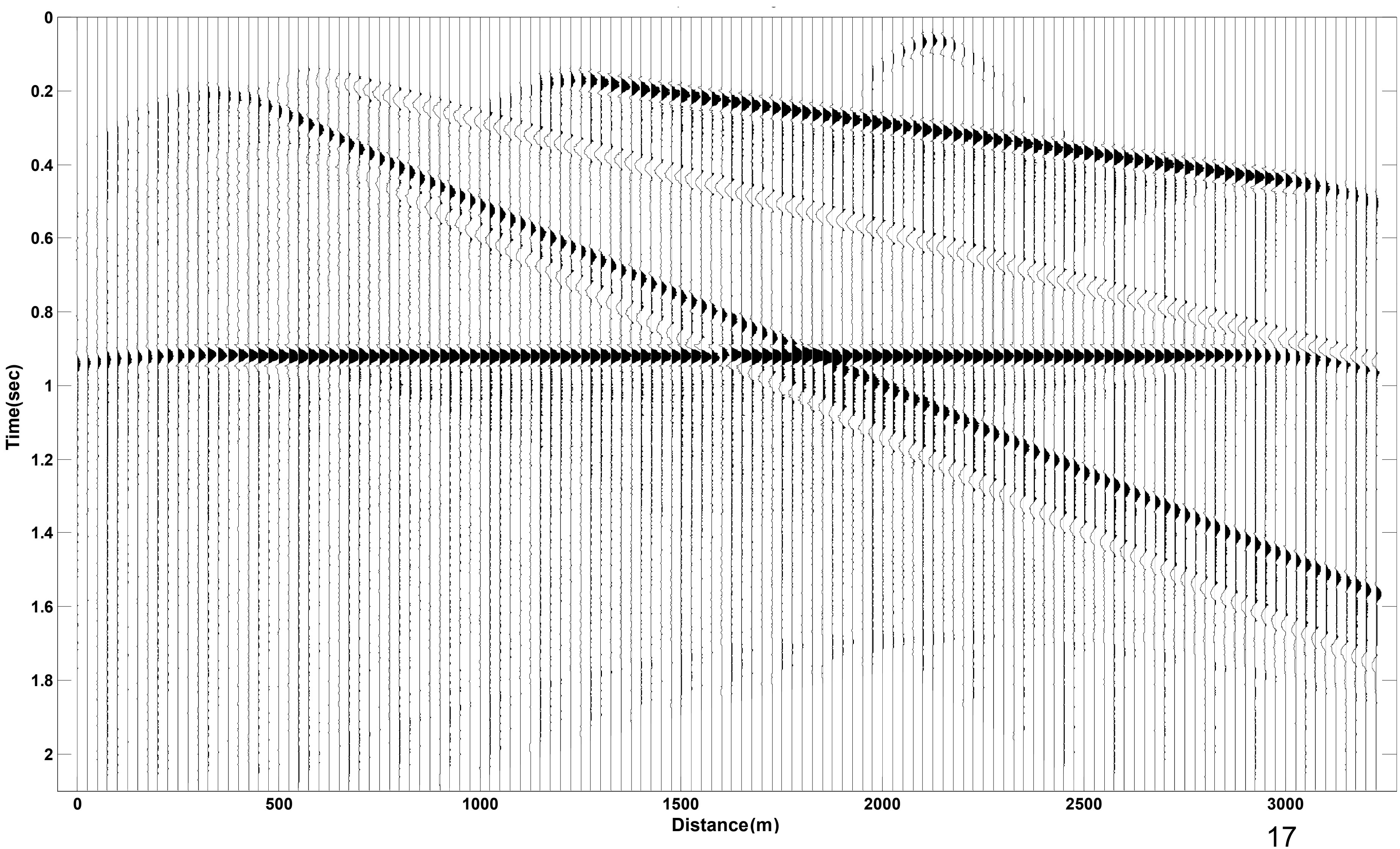
$\mathcal{R}$: Regularization term,

μ : Trade-off parameter

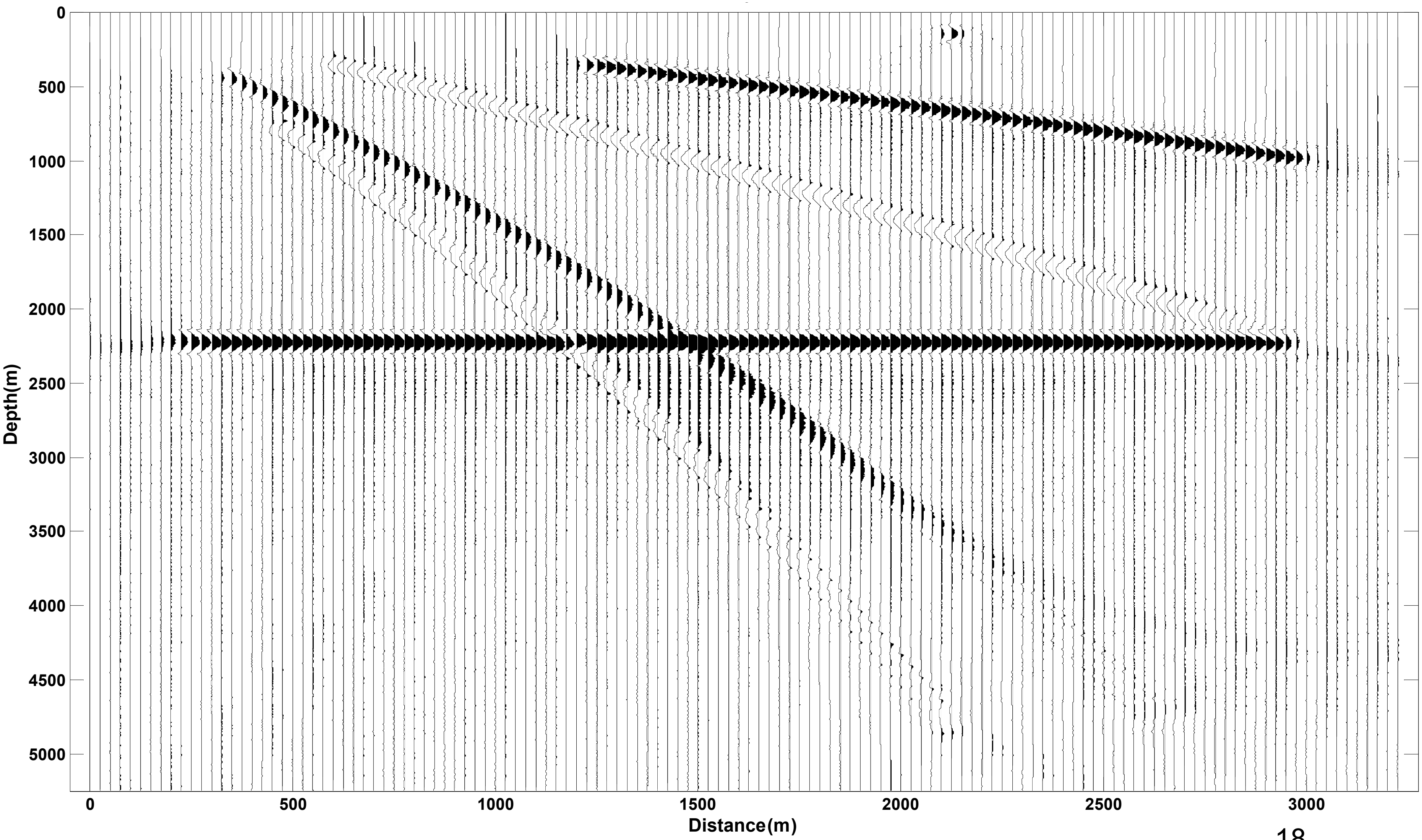
LSM, a synthetic example: Model



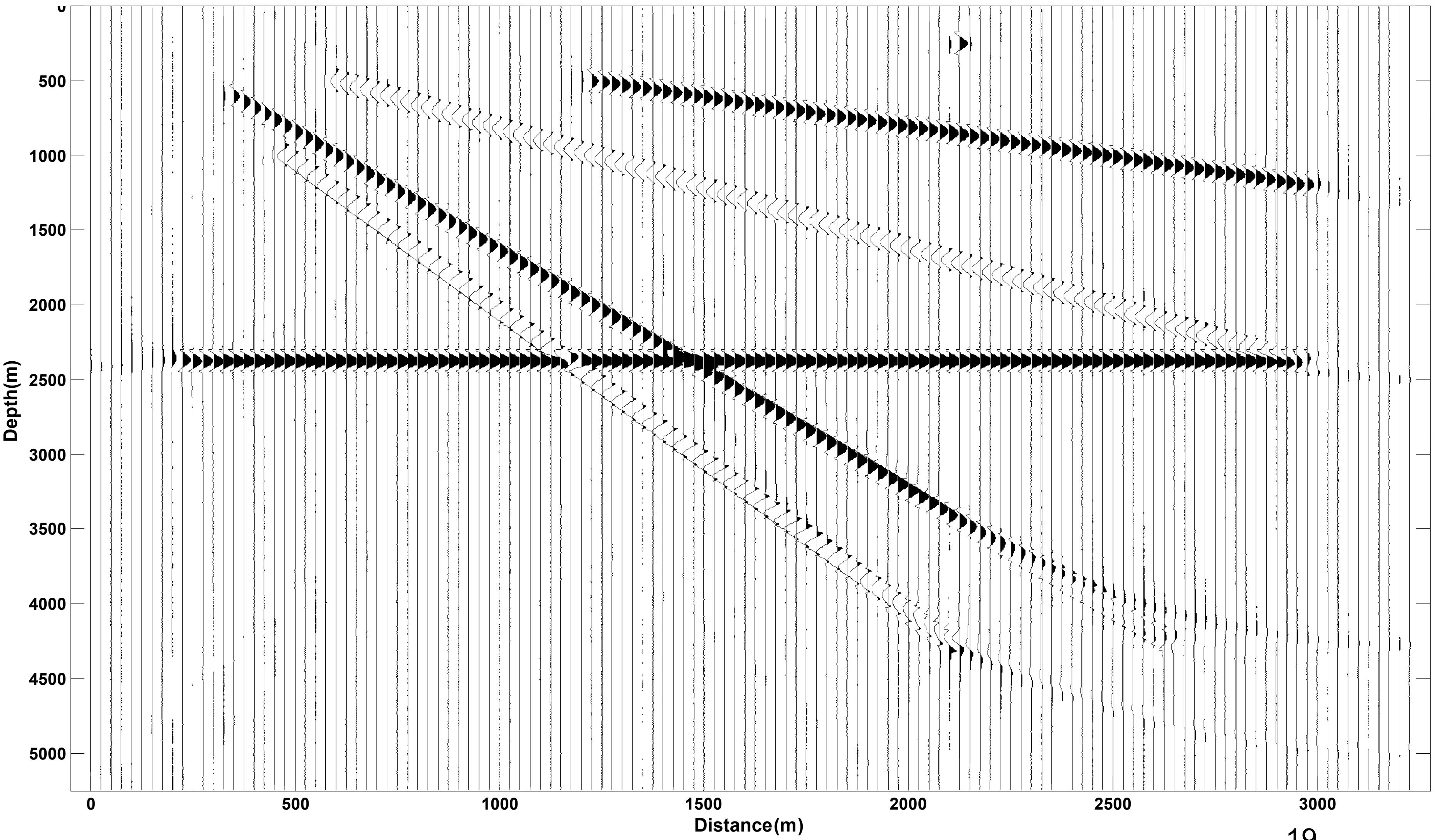
LSM, a synthetic example: Data



LSM, a synthetic example: Migration



LSM, a synthetic example: LSM



LSM, how?

- Least Squares CG is typical solver.

$$(\boldsymbol{G}^T\boldsymbol{G}+\mu^2\boldsymbol{I})\ \ \boldsymbol{m}_{DSL}=\boldsymbol{G}^T\boldsymbol{d}$$

✓ No need to have explicit form for $\boldsymbol{G}^T\ or\ \boldsymbol{G}$

o Converge to the a local minimum which is closest to the initial point.

o Expensive, each iteration = 2 x mig/modeling cost.

✓ Looking for a replacement: Multigrid?

Scales (1978).

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Multigrid, why?

1. Multigrid has been successful, optimal and/or robust for solving many problems, especially PDEs.

2. Faster recovery of low frequencies.

3. Multigrid can converge to the global minimum.

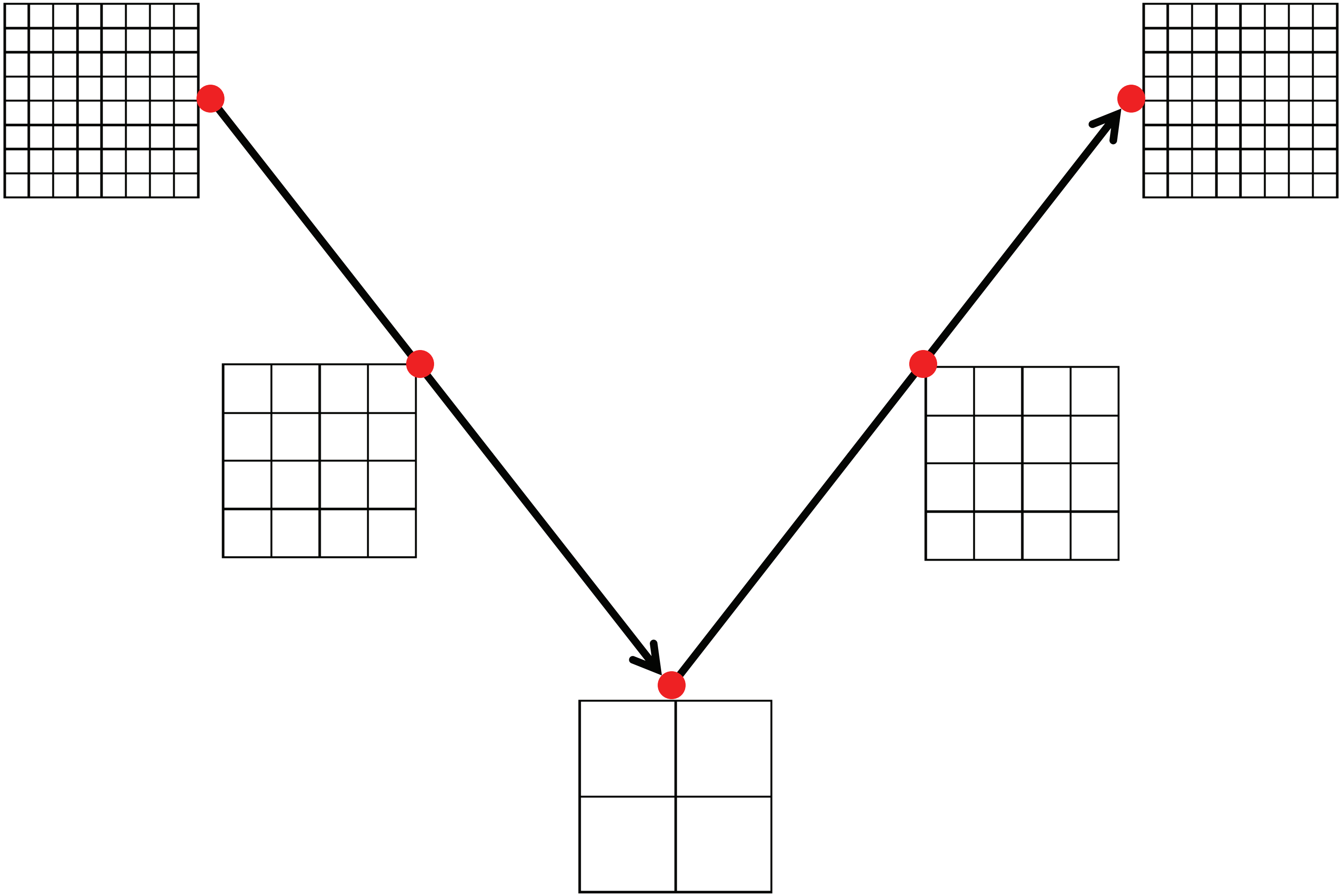
✓ Why not try it on LSM?

Multigrid, what is it?

- Multigrid is not a an iterative method.
- It uses an iterative solver in different grid sizes.

- Solvers: Jacobi/ Gauss-Seidel (G-S).

Multigrid, How does it work? V-cycle



Multigrid, How does Jacobi work?

$$(\boldsymbol{G}^T\boldsymbol{G} + \mu^2\boldsymbol{I}) \quad \boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d} \qquad (1)$$

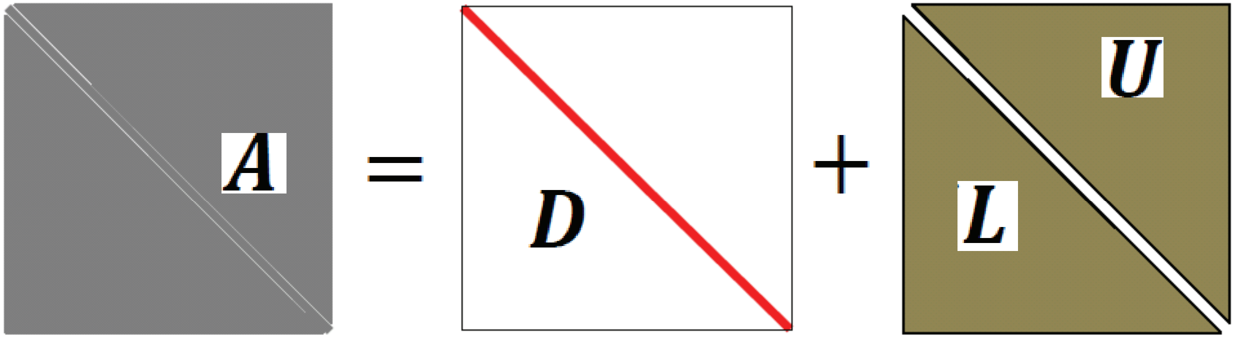
$$\boldsymbol{A} \quad \boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d} \qquad (2)$$

Multigrid, How does Jacobi work?

$$(\boldsymbol{G}^T\boldsymbol{G} + \mu^2\boldsymbol{I}) \quad \boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d} \tag{1}$$

$$\boldsymbol{A} \quad \boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d} \tag{2}$$

$$\boxed{\boldsymbol{A} = \boldsymbol{D} - (\boldsymbol{L} + \boldsymbol{U})} \tag{3}$$

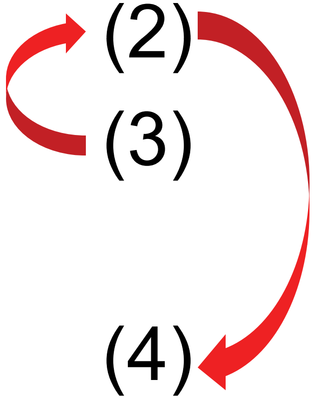


Multigrid, How does Jacobi work?

$$(\boldsymbol{G}^T\boldsymbol{G} + \mu^2\boldsymbol{I}) \quad \boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d} \tag{1}$$

$$\boldsymbol{A} \qquad\qquad \boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d} \tag{2}$$

$$\boxed{\boldsymbol{A} = \boldsymbol{D} - (\boldsymbol{L} + \boldsymbol{U})} \tag{3}$$



$$\boldsymbol{D}\boldsymbol{m}_{DSL} - (\boldsymbol{L} + \boldsymbol{U})\boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d} \tag{4}$$

$$\boldsymbol{D}\boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d} + (\boldsymbol{L} + \boldsymbol{U})\boldsymbol{m}_{DSL} \tag{5}$$

- Jacobi Iteration:

$$\boxed{\boldsymbol{m}_{DSL}^{k+1} = \boldsymbol{D}^{-1}(\boldsymbol{G}^T\boldsymbol{d} + (\boldsymbol{L} + \boldsymbol{U})\boldsymbol{m}_{DSL}^{k})} \tag{6}$$

Multigrid, how does it work?

- Jacobi Iteration:

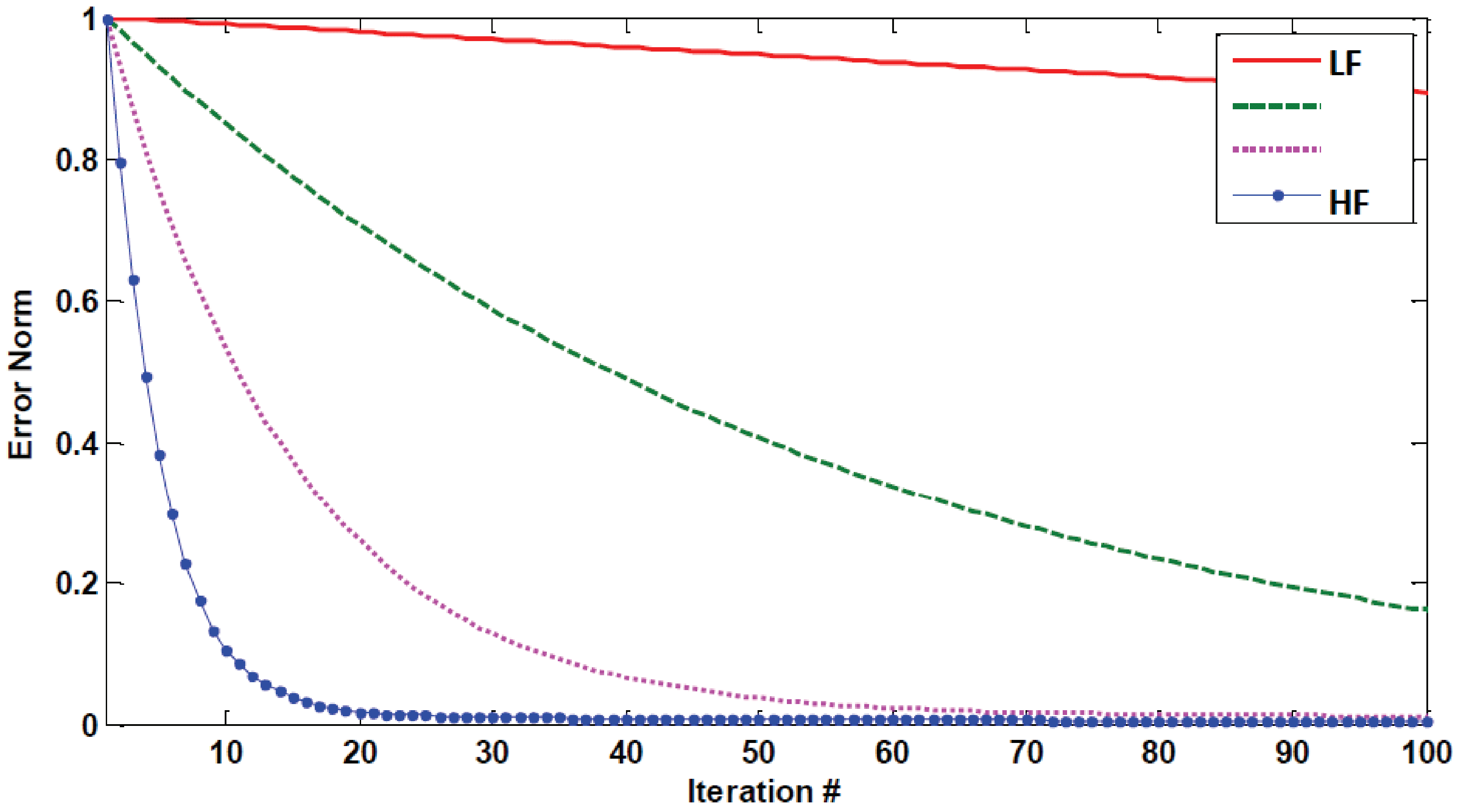
$$\boldsymbol{m}_{DSL}^{k+1} = \boldsymbol{D}^{-1}(\boldsymbol{G}^T\boldsymbol{d} + (\boldsymbol{L} + \boldsymbol{U})\boldsymbol{m}_{DSL}^{k})$$

- Why Jacobi/G-S as the solver?

-Because they are smoothers!

Multigrid, how does it work?

- Convergence of the Jacobi/G-S depend on the frequency content.



Multigrid, how does it work?

- In the multigrid method, we ***Restrict*** problem to a coarser/decimated grid.

- Then we solve equation in coarser/ decimated grid when low frequencies act as high frequencies.

- Then result being ***Interpolated*** to the finer (main) grid size and used as the starting point in the main grid.

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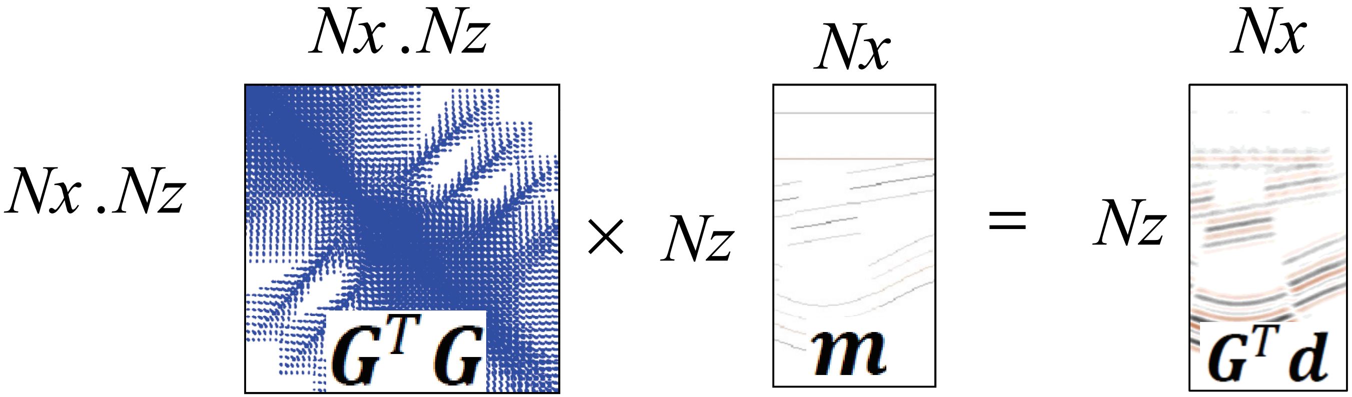
Multigrid, is it applicable to Kirch. LSM?

1. Jacobi & G-S need the explicit form of $\boldsymbol{G}^T\boldsymbol{G}$ matrix. The Kirchhoff $\boldsymbol{G}^T\boldsymbol{G}$ matrix is large and non sparse; $\boldsymbol{G}^T\boldsymbol{G}\,\boldsymbol{m} = \boldsymbol{G}^T\boldsymbol{d}$



Example: 2D line, Gulf of Mexico: ***79 TB***

$$(Nx.Nz)^2 = (1800\ \times 1750)^2 > 9.9 \times 10^{12}$$

Multigrid, is it applicable to Kirch. LSM?

2. Jacobi & G-S need a diagonally dominant matrix :

$$|diagonal| \geq \sum |Non - diagonal|$$

$$\boldsymbol{m}_{DSL}^{k+1} = \boldsymbol{D}^{-1}(\boldsymbol{G}^T\boldsymbol{d} + (\boldsymbol{L} + \boldsymbol{U})\boldsymbol{m}_{DSL}^{k})$$

Many examples show that our $\boldsymbol{G}^T\boldsymbol{G}$ matrix is not diagonally dominant.

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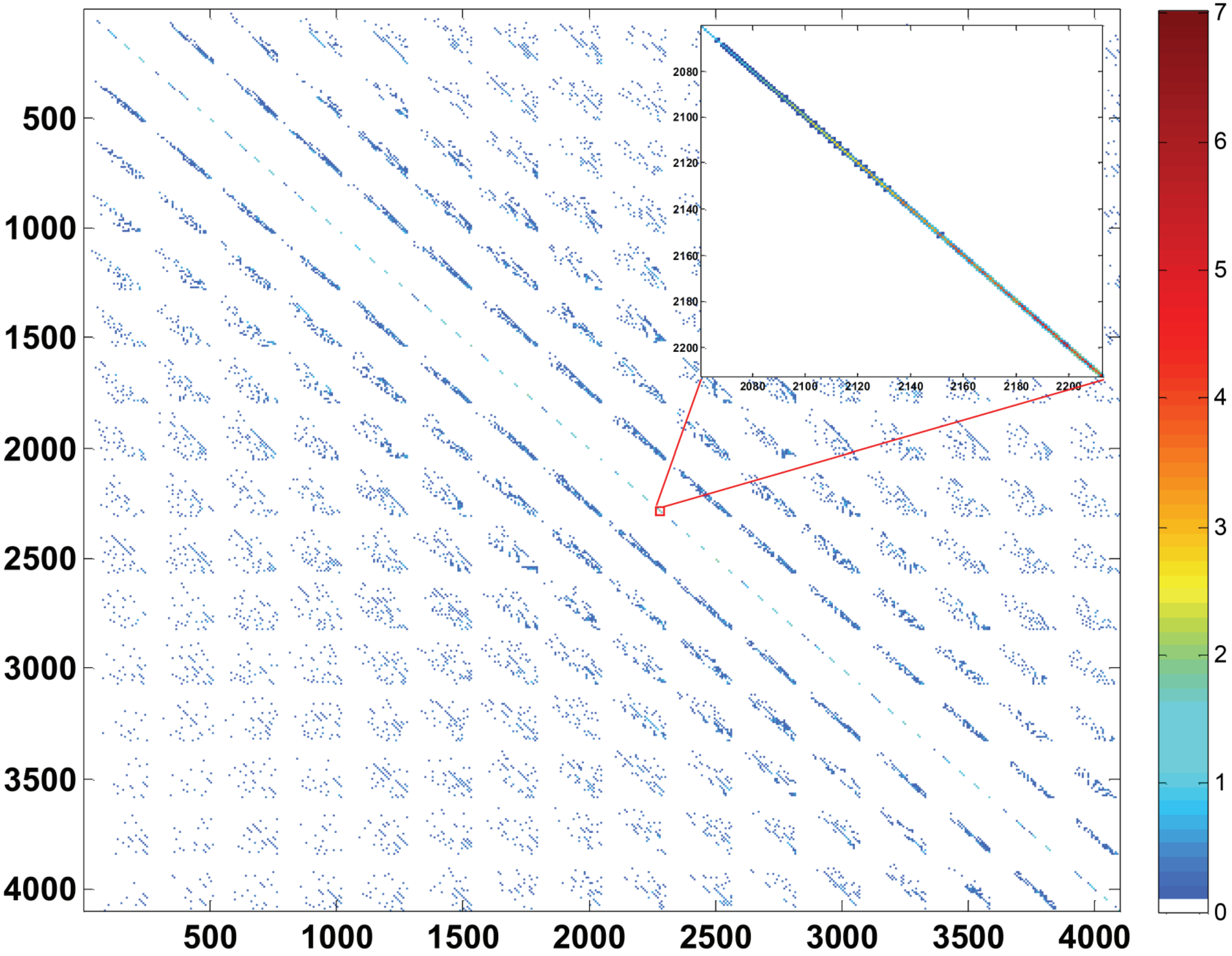
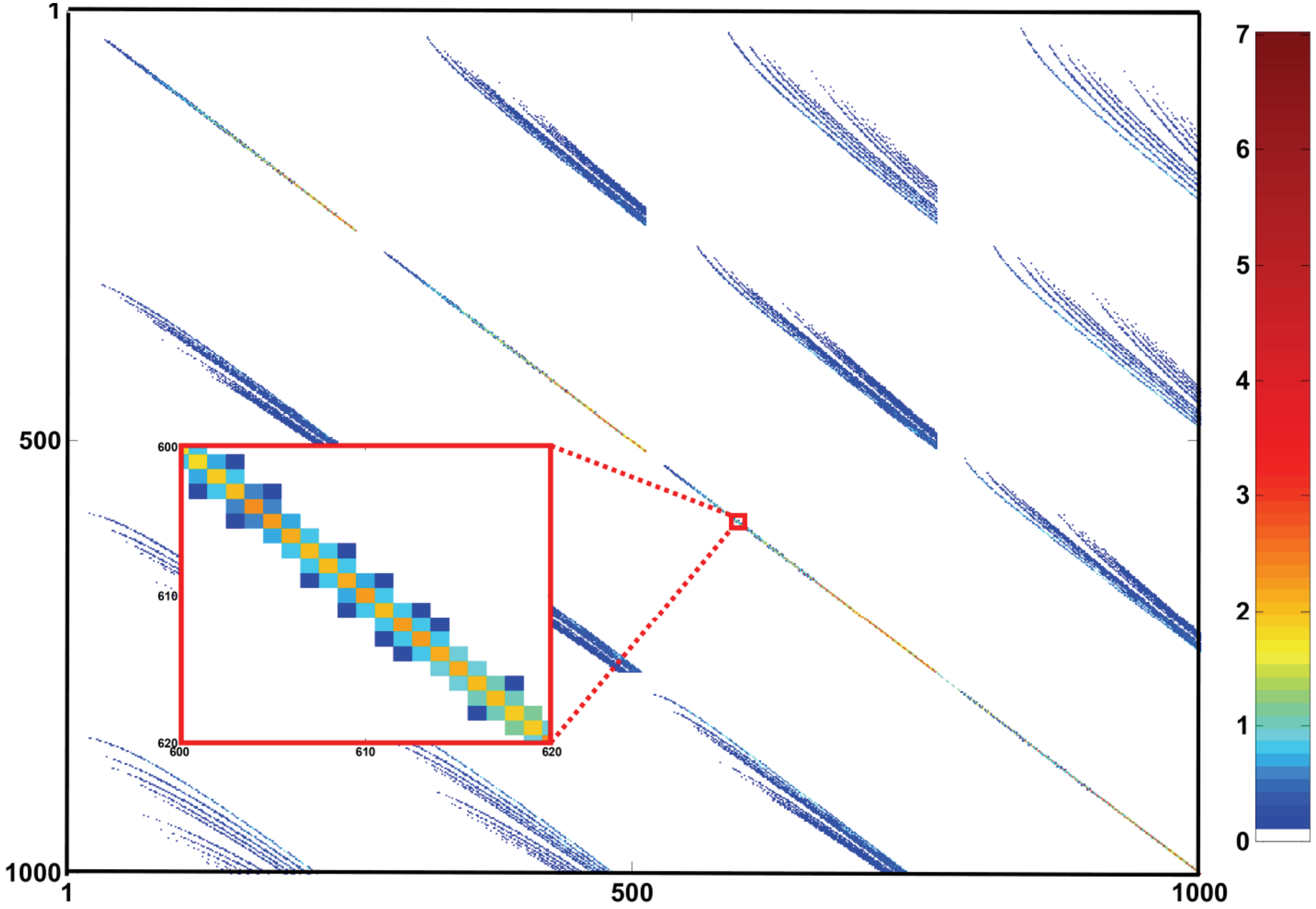
Multigrid, problem with $G^T G$

Example: $(G^T G + \mu^2 I) \;\; m_{DSL} = G^T d$



Multigrid, problem with $\boldsymbol{G}^T\boldsymbol{G}$

Example: $(\boldsymbol{G}^T\boldsymbol{G} + \mu^2\boldsymbol{I})\ \ \boldsymbol{m}_{DSL} = \boldsymbol{G}^T\boldsymbol{d}$



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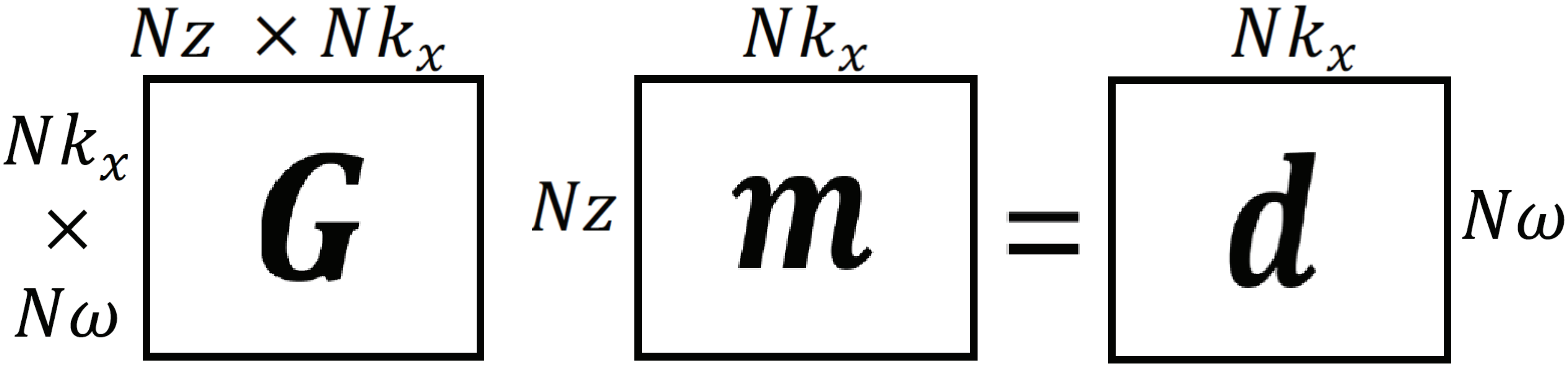


$$\begin{array}{cc} & Nz \times Nk_x \\ \begin{array}{c} Nk_x \\ \times \\ N\omega \end{array} & \boxed{\boldsymbol{G}} \end{array} \begin{array}{cc} & Nk_x \\ Nz & \boxed{\boldsymbol{m}} \end{array} = \begin{array}{cc} Nk_x & \\ \boxed{\boldsymbol{d}} & N\omega \end{array}$$

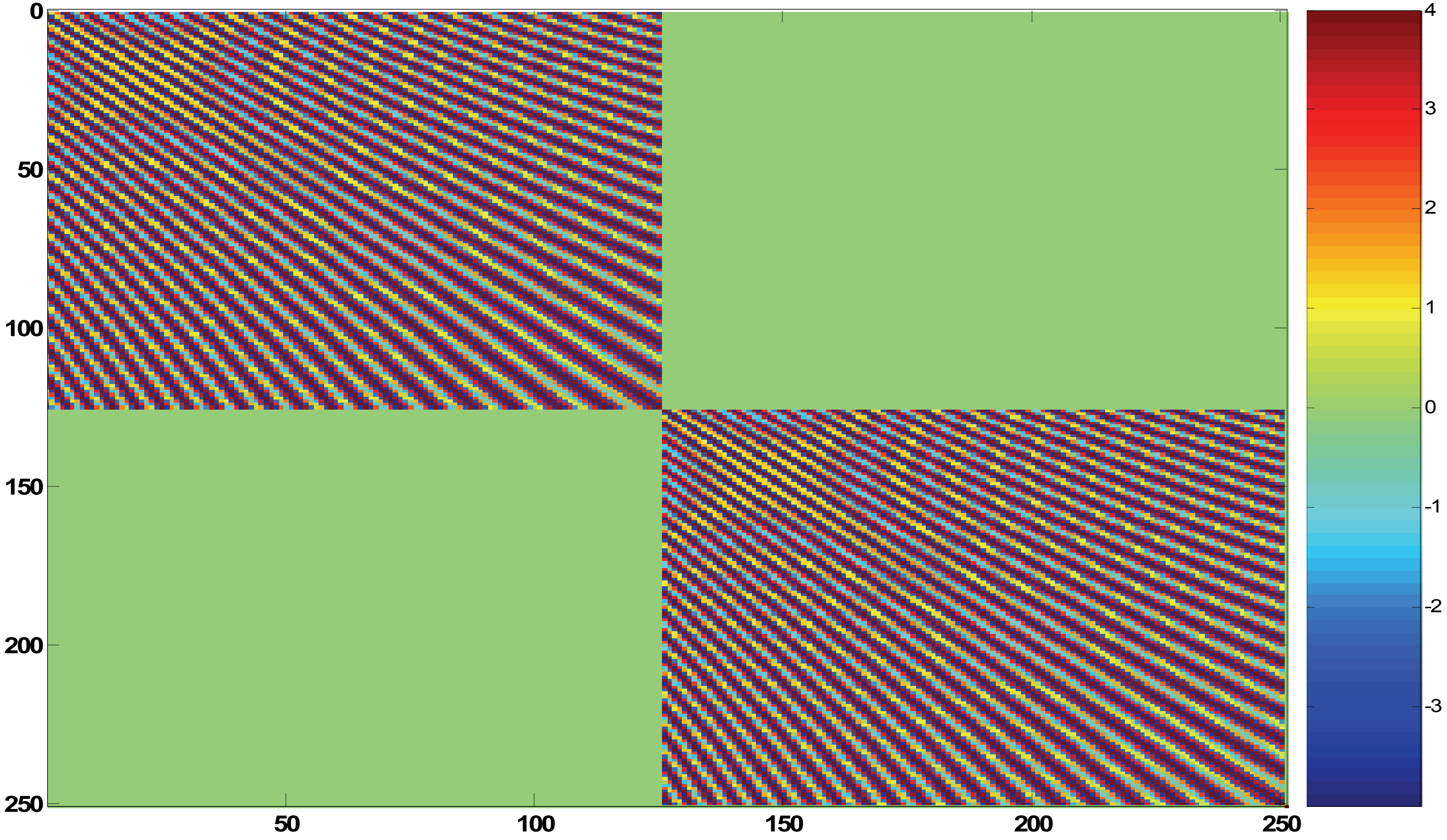
Gazdag LSM, 1st scenario:

LSM with whole migration /modeling operators:

- Size $\boldsymbol{G}'\boldsymbol{G}+\mu^2\boldsymbol{I}$:$Nz \times Nk_x \text{by } Nz \times Nk_x$
- Matrix is dense and large.
- Matrix is not diagonally dominant.

\+ One equation to be solved.

Gazdag LSM, 1^{st} scenario:



Gazdag LSM, 2nd scenario:

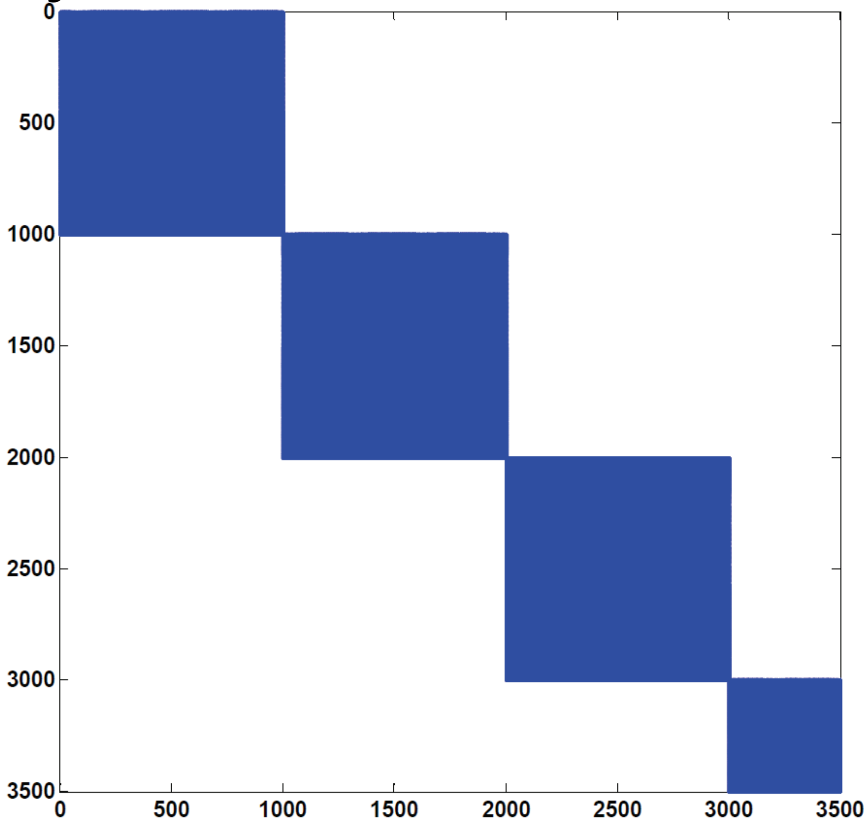
LSM for each temporal frequency:

-Size $\boldsymbol{G}'\boldsymbol{G}+\mu^2\boldsymbol{I}$: $Nz \times Nk_x$ by $Nz \times Nk_x$

-Matrix is not diagonally dominant.

- $N\omega$ equations

 to be solved.



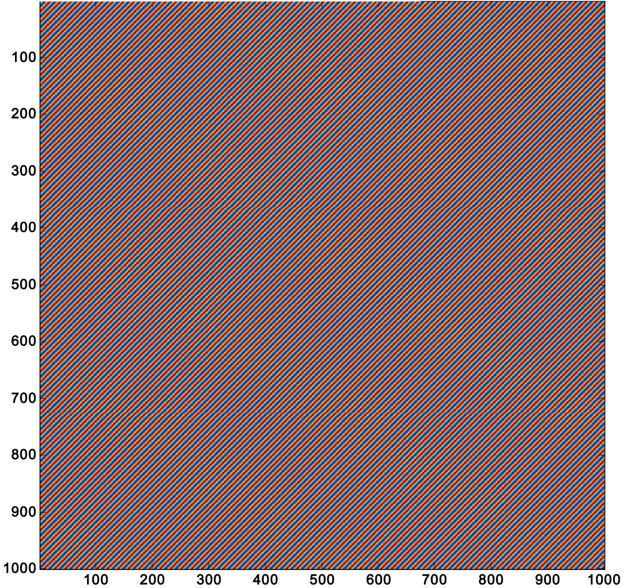
Gazdag LSM, 3rd scenario:

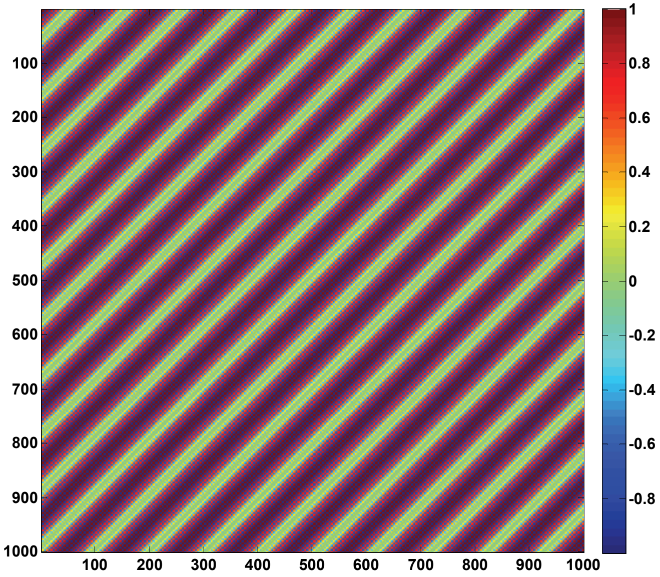
LSM for each temporal and spatial frequency:

+Size $\boldsymbol{G'G} + \mu^2\boldsymbol{I}$: $Nz \times Nz$

-Matrix is not diagonally dominant, dense

-$N\omega \times Nk_x$ equations to be solved.





Gazdag LSM, 4th scenario:

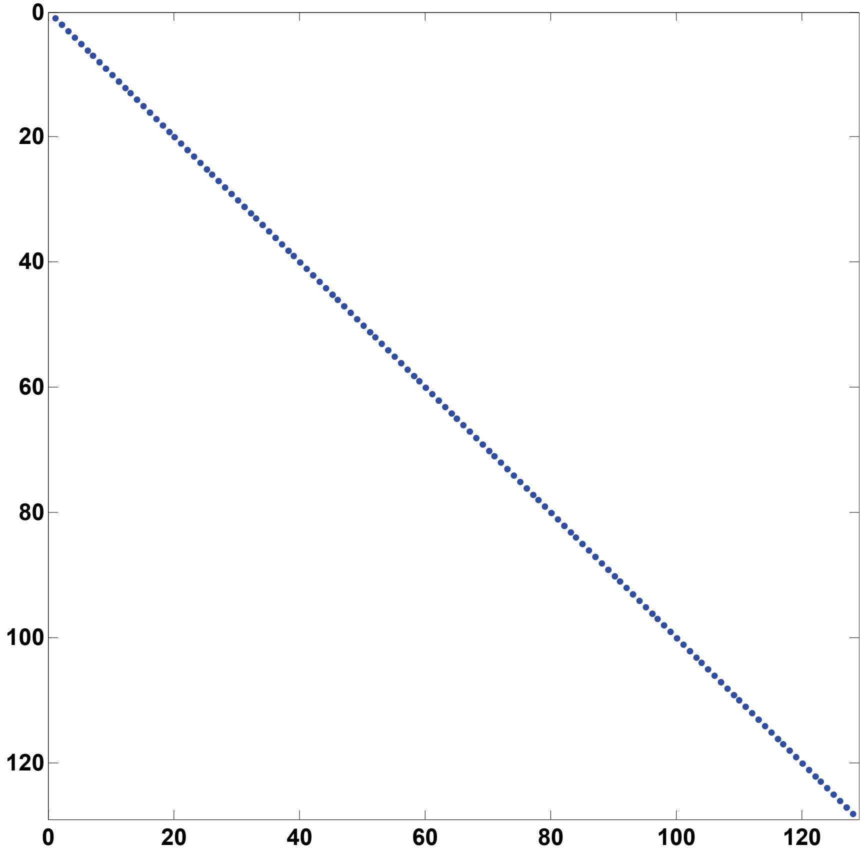
LSM for each temporal frequency and depth:

+Size $\boldsymbol{G}'\boldsymbol{G} + \mu^2\boldsymbol{I}$: $Nk_x \times Nk_x$

+A diagonal Matrix.

-$N\omega \times Nz$ equations

to be solved.



Gazdag LSM, 4th scenario:

LSM for each temporal frequency and depth:

- It is easier to add some constraint to the problem
- Some constraints make it non-diagonal, and Jacobi/multigrid methods can be used to solve the problem faster.

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Summary

- Conventional multigrid which uses Jacobi or Gauss-Seidel is not viable to solve Kirchhoff LSM problem for two reasons:
    1. Hessian matrix is too large.
    2. Hessian matrix of the LSM is not diagonally dominant and solvable with Jacobi or G-S.
- CG is an effective solver, but it is not a smoother. Same result, more expensive.

Summary

- Same problems arise when applying multigrid to solve Gazdag LSM.
- However, the size of the $\boldsymbol{G}'\boldsymbol{G} + \mu^2\boldsymbol{I}$ matrix can be reduced by implementing LSM for each temporal and spatial frequency or depth.
- Reducing this size is equivalent to the increasing the number of LSM to be solved.

Summary

- By implementing LSM for each temporal and spatial frequency $\boldsymbol{G}'\boldsymbol{G} + \mu^2\boldsymbol{I}$ is still a diagonally non-dominant matrix, therefore multigrid is not optimum available solver.
- In the constrained LSM for each temporal frequency and depth, multigrid may be faster than LSCG.

Future Work

This is an ongoing research.

Trying with the other constraints.

Applying multigrid on the prestack phase shift migration.

Acknowledgment

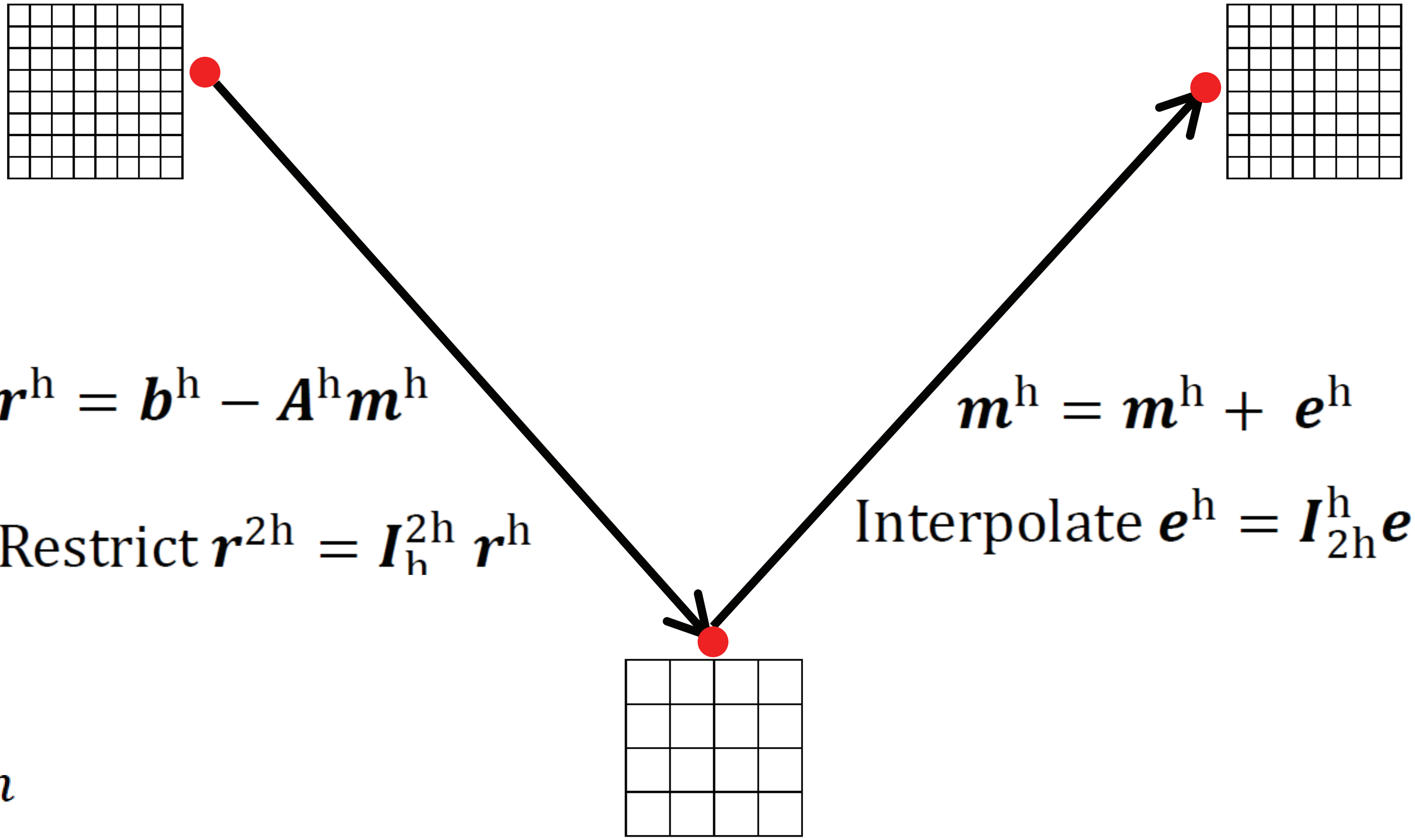
- I would like to thank:
  - Dr. John C. Bancroft.
  - Dr. Margrave, Dr. Ferguson, Dr. Naghizadeh, M. Wilson.
  - CREWES faculty members, sponsors and NSERC.
  - Thank you.

?

How does it work? v-cycle

$$\boldsymbol{G}^T \boldsymbol{G}\, \boldsymbol{m} = \boldsymbol{G}^T \boldsymbol{d}$$

$$\boldsymbol{A}\boldsymbol{m} = \boldsymbol{b}$$



Solve $\boldsymbol{A}^{\mathrm{h}}\boldsymbol{m}^{\mathrm{h}} = \boldsymbol{b}^{h}$ to have $\boldsymbol{m}^{\mathrm{h}}$

Solve $\boldsymbol{A}^{\mathrm{h}}\boldsymbol{m}^{\mathrm{h}} = \boldsymbol{b}^{\mathrm{h}}$ with $\boldsymbol{m}^{\mathrm{h}}$ as initial

$\boldsymbol{m}^{h}$

$\boldsymbol{r}^{\mathrm{h}} = \boldsymbol{b}^{\mathrm{h}} - \boldsymbol{A}^{\mathrm{h}}\boldsymbol{m}^{\mathrm{h}}$

Restrict $\boldsymbol{r}^{2\mathrm{h}} = \boldsymbol{I}_{\mathrm{h}}^{2\mathrm{h}}\, \boldsymbol{r}^{\mathrm{h}}$

$\boldsymbol{m}^{\mathrm{h}} = \boldsymbol{m}^{\mathrm{h}} + \boldsymbol{e}^{\mathrm{h}}$

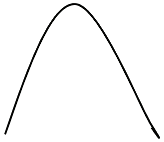
Interpolate $\boldsymbol{e}^{\mathrm{h}} = \boldsymbol{I}_{2\mathrm{h}}^{\mathrm{h}}\boldsymbol{e}^{2\mathrm{h}}$

$\boldsymbol{m}^{2h}$

Solve $\boldsymbol{A}^{2\mathrm{h}}\boldsymbol{e}^{2\mathrm{h}} = \boldsymbol{r}^{2\mathrm{h}}$ to have $\boldsymbol{e}^{2\mathrm{h}}$

LSCG and synthetic examples

```
for $i = 1$: # of iterations
    $\alpha_{i+1} = \frac{\boldsymbol{r}_i . \boldsymbol{r}_i}{\boldsymbol{q}_i . \boldsymbol{q}_i}$
    $\boldsymbol{m}_{i+1} = \boldsymbol{m}_i + \alpha_{i+1}\boldsymbol{p}_i$
    $\boldsymbol{s}_{i+1} = \boldsymbol{s}_i - \alpha_{i+1}\boldsymbol{q}_i$
    $\boldsymbol{r}_{i+1} = \boldsymbol{G}'\boldsymbol{s}_{i+1}$        *
    $\beta_{i+1} = \frac{\boldsymbol{r}_{i+1} . \boldsymbol{r}_{i+1}}{\boldsymbol{r} . \boldsymbol{r}}$
    $\boldsymbol{p}_{i+1} = \boldsymbol{r} + \beta\boldsymbol{p}_i$
    $\boldsymbol{q}_{i+1} = \boldsymbol{G}\boldsymbol{p}_{i+1}$        **
    $i = i + 1$
endfor
```



Why Multigrid? **Jacobi/Gauss-Seidel**

- Why Jacobi? **Because of its smoothing property.**

$$\boldsymbol{v}^{k+1} = \boldsymbol{D}^{-1}(\boldsymbol{L}+\boldsymbol{U})\,\boldsymbol{v}^{k} + \boldsymbol{D}^{-1}\boldsymbol{b}$$

$$v_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij} v_j^k \right), i = 1,2,\ldots.N$$

$$\mathbf{Ax} = \mathbf{0}$$

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & \cdots & \\ 0 & -1 & 2 & -1 & & \\ & \vdots & & & \ddots & \vdots \\ & & & & \cdots & 2 \end{bmatrix}.$$

II. Multigrid

- Example: Solve $\boldsymbol{Ax = 0}$

where

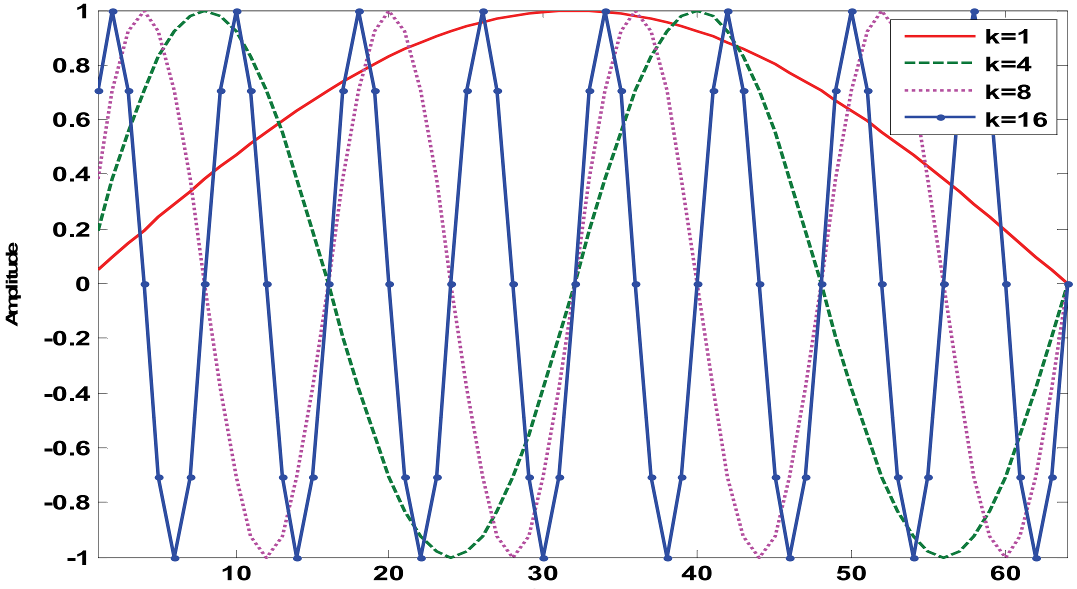
$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & \cdots & \\ 0 & -1 & 2 & -1 & & \\ & & \vdots & & \ddots & \vdots \\ & & & & \cdots & 2 \end{bmatrix}.$$

Lets start with these initial values:

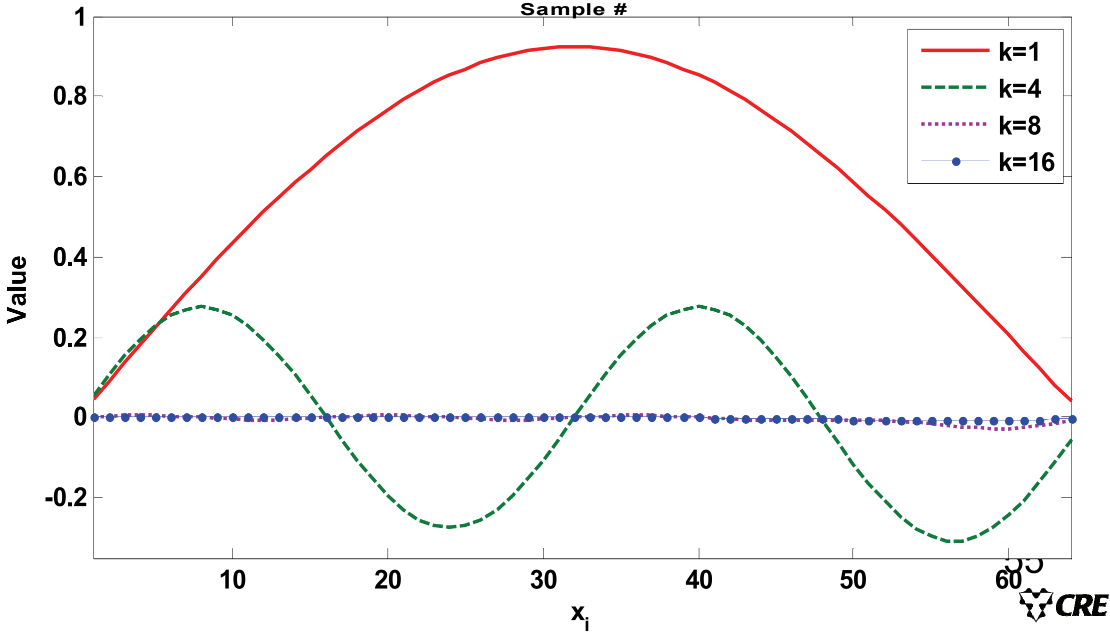
$$\boldsymbol{x0_j = sin\left(\frac{jk\pi}{n}\right), \quad 0 \leq j \leq n\,, 1 \leq k \leq n-1\,, n = 64}$$

II. Multigrid

- Starting values:

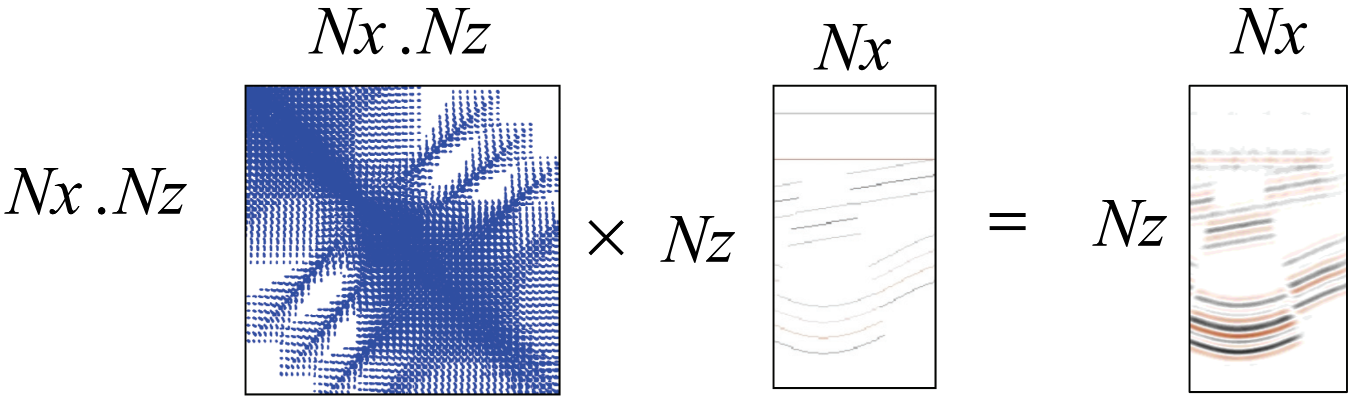


- Solution:



Multigrid? Problem with size

$$\mathbf{G'\,G\,\hat{m} = G'\,d}$$



$$Nx\,.Nz\,.Nx\,.Nz\, = 1800\, * 1750\, * 1800\, * 1750\, \approx 9.92\times 10^{12}$$

II. Multigrid Problem with $G^T G$

Example: $(\boldsymbol{G}^T \boldsymbol{G} + \mu^2 \boldsymbol{I}) \ \ \boldsymbol{m}_{DSL} = \boldsymbol{G}^T \boldsymbol{d}$