FWI model space coordinate system design based on data misfit

CREWES Annual Sponsors' Meeting

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Summary

When we:

- Discretize a computational volume and select parameter classes in FWI
- Parameterize AVO in terms of (I_P, I_S, ρ), instead of (V_P, V_S, ρ)

…we are overlaying a coordinate system onto model space.

When we change that parameterization, we are transforming from one coordinate system to another.

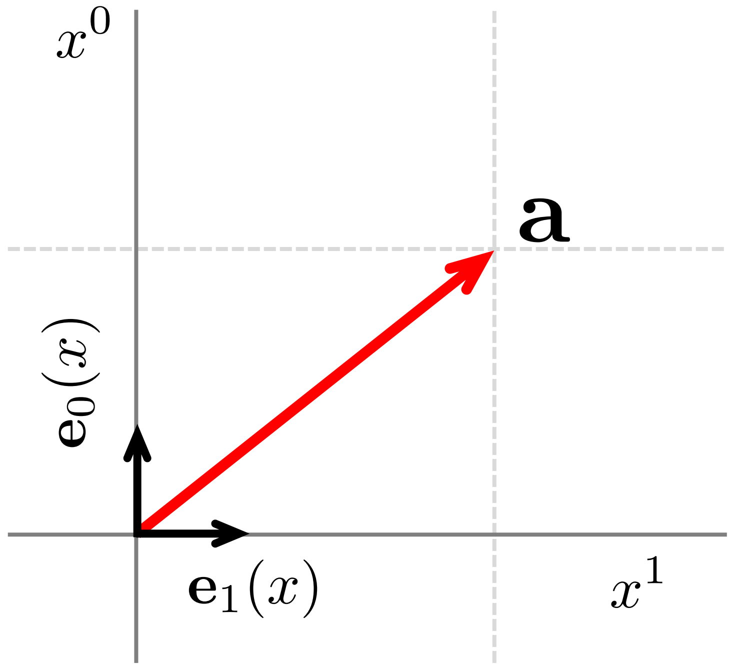
This can be made precise using tensor analysis. E.g., (I_P, I_S, ρ)→(V_P, V_S, ρ) is a transform between contravariant vectors in oblique/rectilinear coordinate systems

We can use this idea to seek new parameterizations. Here we focus on a class of transformations to systems with favourable optimization properties.

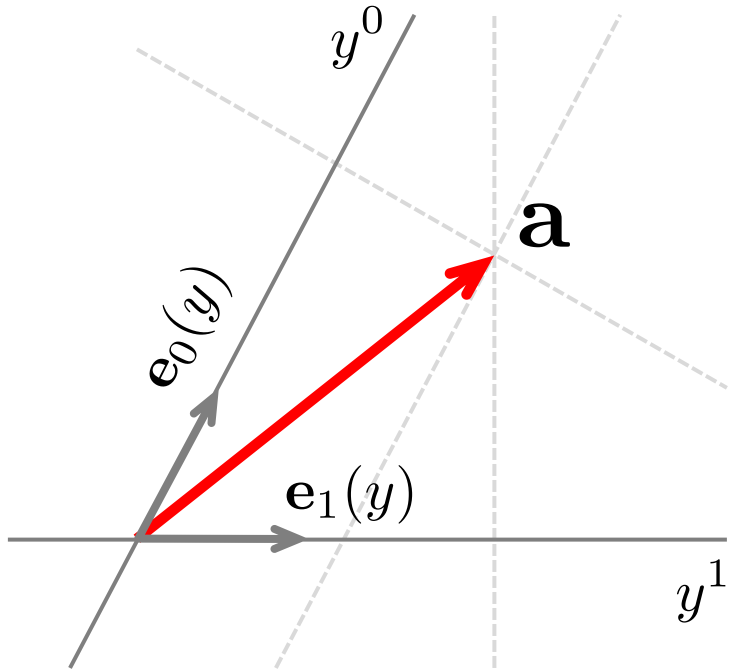
Plan:

- Confirm that FWI re-parameterization ↔ oblique-rectilinear coordinate transformation
- Formulate a coordinate transform problem constrained by NxN Hessian operators
- Investigate consequences for inversion

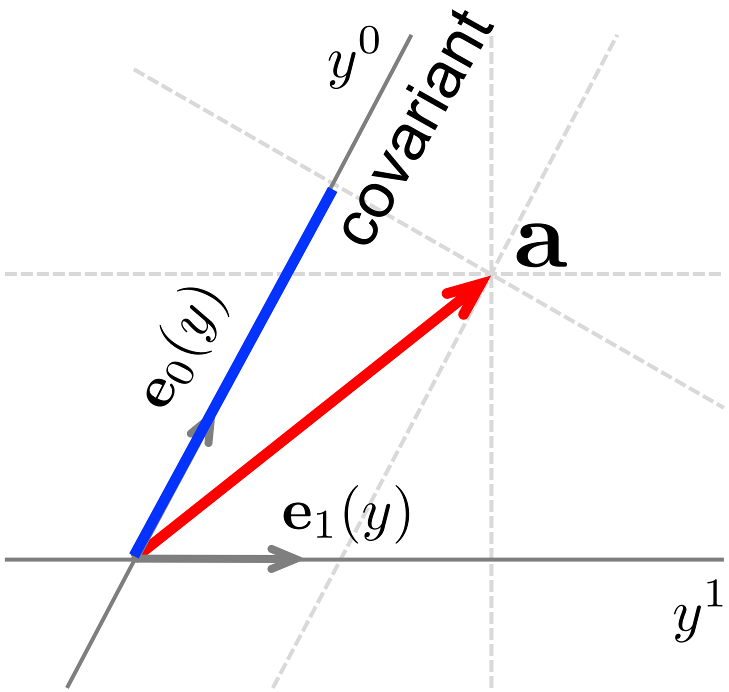
Vectors and tensors in non-Cartesian coordinate systems

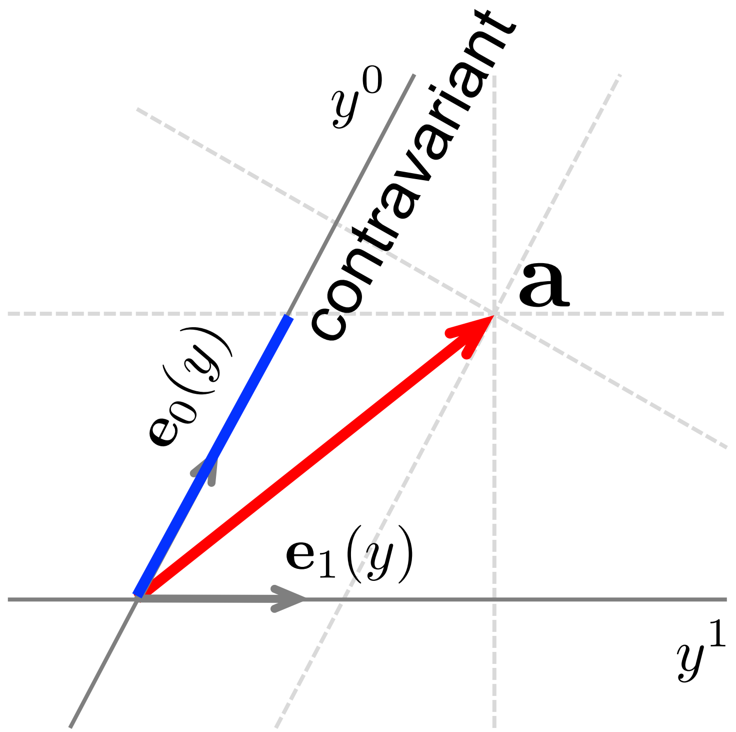


Vectors and tensors in non-Cartesian coordinate systems

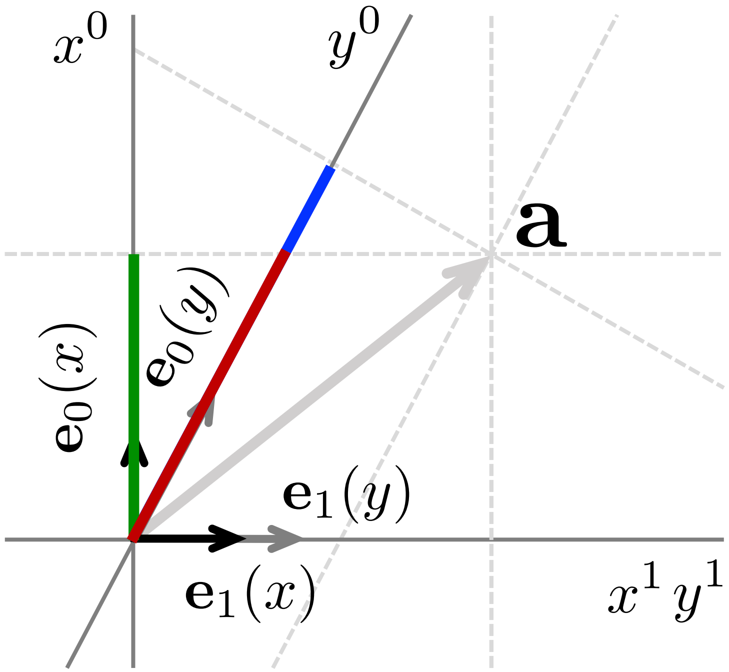


Vectors and tensors in non-Cartesian coordinate systems

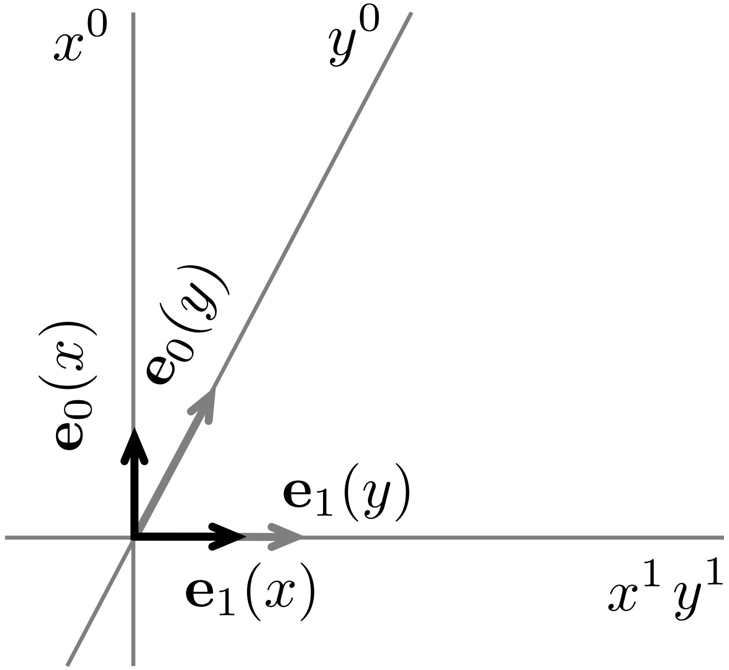



contravariant
y^0
$\mathbf{a}$
$\mathbf{e}_0(y)$
$\mathbf{e}_1(y)$
y^1

Vectors and tensors in non-Cartesian coordinate systems



Vectors and tensors in non-Cartesian coordinate systems



$$\begin{bmatrix} t^0_0 & t^0_1 \\ t^1_0 & t^1_1 \end{bmatrix}$$

$\mathbf{e}_0(y)$ $\mathbf{e}_1(y)$

y to x, contravariant

Vectors and tensors in non-Cartesian coordinate systems

$$\begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix} \qquad \begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^{-1} \qquad \left(\begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^{-1}\right)^T \qquad \begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^T$$

transformation of contravariant components from y to x

Vectors and tensors in non-Cartesian coordinate systems

$$\begin{bmatrix} t^0_0 & t^0_1 \\ t^1_0 & t^1_1 \end{bmatrix} \qquad \begin{bmatrix} t^0_0 & t^0_1 \\ t^1_0 & t^1_1 \end{bmatrix}^{-1} \qquad \left(\begin{bmatrix} t^0_0 & t^0_1 \\ t^1_0 & t^1_1 \end{bmatrix}^{-1}\right)^T \qquad \begin{bmatrix} t^0_0 & t^0_1 \\ t^1_0 & t^1_1 \end{bmatrix}^T$$

transformation of contravariant components from x to y

Vectors and tensors in non-Cartesian coordinate systems

$$\begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix} \qquad \begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^{-1} \qquad \left(\begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^{-1}\right)^T \qquad \begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^T$$

transformation of covariant components from y to x

$$\begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix} \qquad \begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^{-1} \qquad \left(\begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^{-1} \right)^T \qquad \begin{bmatrix} t_0^0 & t_1^0 \\ t_0^1 & t_1^1 \end{bmatrix}^T$$

transformation of covariant components from x to y

Model re-parameterization as a coordinate transform

Assertion:

- All model space re-parameterizations are coordinate transforms between oblique, rectilinear coordinate systems

To justify the assertion:

- Show that it is true in several very different examples, then generalize
- Approach: assume model (“position”) vectors to be contravariant, then see if gradients of a scalar function of these vectors transform as covariant vectors.

True for:

- All AVO re-parameterizations
- Changes in FWI parameter classes (e.g., velocity-density to modulus-density)
- Model space preconditioning (e.g., Harlan, Fomel, Claerbout)

Show now:

- Changes in FWI parameter classes

Model re-parameterization as a coordinate transform

$$s_\kappa = \frac{1}{\kappa}$$

$$[\partial_x, \partial_z] \cdot \left(s_\rho \begin{bmatrix} \partial_x u \\ \partial_z u \end{bmatrix} \right) + \omega^2 s_\kappa u = f \longrightarrow \mathcal{A}\mathbf{u} = \mathbf{f}$$

$$s_\rho = \frac{1}{\rho}$$

	x_1	x_2
z_1	1 u, s_ρ, s_κ	2 u, s_ρ, s_κ
z_2	3 u, s_ρ, s_κ	4 u, s_ρ, s_κ

$$\omega^2 s_{\kappa_3} - 2s_{\rho_3} - s_{\rho_4}$$

$$\mathcal{A}\mathbf{u} = \mathbf{f} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

Model re-parameterization as a coordinate transform

$$s^\mu = \begin{bmatrix} s^1 \\ s^2 \\ s^3 \\ s^4 \\ s^5 \\ s^6 \\ s^7 \\ s^8 \end{bmatrix} = \begin{bmatrix} s_{\rho_1} \\ s_{\rho_2} \\ s_{\rho_3} \\ s_{\rho_4} \\ s_{\kappa_1} \\ s_{\kappa_2} \\ s_{\kappa_3} \\ s_{\kappa_4} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} \omega^2 s^5 - 2s^1 - s^2 - s^3 & s^2 & s^3 & 0 \\ s^2 & \omega^2 s^6 - 2s^2 - s^4 & 0 & s^4 \\ s^3 & 0 & \omega^2 s^7 - 2s^3 - s^4 & s^4 \\ 0 & s^4 & s^4 & \omega^2 s^8 - 2s^4 \end{bmatrix}}_{a_{ij}}$$

Model re-parameterization as a coordinate transform

$$s^\mu = \begin{bmatrix} s^1 \\ s^2 \\ s^3 \\ s^4 \\ s^5 \\ s^6 \\ s^7 \\ s^8 \end{bmatrix} = \begin{bmatrix} s_{\rho_1} \\ s_{\rho_2} \\ s_{\rho_3} \\ s_{\rho_4} \\ s_{\kappa_1} \\ s_{\kappa_2} \\ s_{\kappa_3} \\ s_{\kappa_4} \end{bmatrix}$$

$$a_{ij}(s) = \begin{bmatrix} \omega^2 s^5 - 2s^1 - s^2 - s^3 & s^2 & s^3 & 0 \\ s^2 & \omega^2 s^6 - 2s^2 - s^4 & 0 & s^4 \\ s^3 & 0 & \omega^2 s^7 - 2s^3 - s^4 & s^4 \\ 0 & s^4 & s^4 & \omega^2 s^8 - 2s^4 \end{bmatrix}$$

Model re-parameterization as a coordinate transform

$$r^\mu = \begin{bmatrix} r^1 \\ r^2 \\ r^3 \\ r^4 \\ r^5 \\ r^6 \\ r^7 \\ r^8 \end{bmatrix} = \begin{bmatrix} r_{\rho_1} \\ r_{\rho_2} \\ r_{\rho_3} \\ r_{\rho_4} \\ r_{c_1} \\ r_{c_2} \\ r_{c_3} \\ r_{c_4} \end{bmatrix}$$

$$a_{ij}(r) = \begin{bmatrix} \omega^2 r^1 r^5 - 2r^1 - r^2 - r^3 & r^2 & r^3 & 0 \\ r^2 & \omega^2 r^2 r^6 - 2r^2 - r^4 & 0 & r^4 \\ r^3 & 0 & \omega^2 r^3 r^7 - 2r^3 - r^4 & r^4 \\ 0 & r^4 & r^4 & \omega^2 r^4 r^8 - 2r^4 \end{bmatrix}$$

Model re-parameterization as a coordinate transform

$$\Delta \mathbf{s} \to \Delta s^{\mu}$$

contravariant

$$\frac{\partial \Phi}{\partial \mathbf{s}} = \mathbf{u}^T \left(\frac{\partial \mathcal{A}}{\partial \mathbf{s}} \right)^T \mathbf{v} \;\to\; \Phi_{,\mu}(s) = u_i(s) \left(\frac{\partial a_{ij}(s)}{\partial s^{\mu}} \right) v_j(s)$$

scalar

covariant?

scalar

Model re-parameterization as a coordinate transform

$$s^\mu = \begin{bmatrix} s^1 \\ s^2 \\ s^3 \\ s^4 \\ s^5 \\ s^6 \\ s^7 \\ s^8 \end{bmatrix} = \begin{bmatrix} s_{\rho_1} \\ s_{\rho_2} \\ s_{\rho_3} \\ s_{\rho_4} \\ s_{\kappa_1} \\ s_{\kappa_2} \\ s_{\kappa_3} \\ s_{\kappa_4} \end{bmatrix}$$

$$a_{ij}(s) = \begin{bmatrix} \omega^2 s^5 - 2s^1 - s^2 - s^3 & s^2 & s^3 & 0 \\ s^2 & \omega^2 s^6 - 2s^2 - s^4 & 0 & s^4 \\ s^3 & 0 & \omega^2 s^7 - 2s^3 - s^4 & s^4 \\ 0 & s^4 & s^4 & \omega^2 s^8 - 2s^4 \end{bmatrix}$$

Model re-parameterization as a coordinate transform

$$\frac{\partial a_{33}}{\partial s^\mu} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \\ 0 \\ 0 \\ \omega^2 \\ 0 \end{bmatrix} \quad \leftrightarrow \quad \frac{\partial a_{33}}{\partial r^\mu} = \begin{bmatrix} 0 \\ 0 \\ \omega^2 r^7 - 2 \\ -1 \\ 0 \\ 0 \\ \omega^2 r^3 \\ 0 \end{bmatrix}$$

covariant?

Model re-parameterization as a coordinate transform

$$\Delta s_\rho = \Delta r_\rho \qquad \Delta s_\kappa = r_c \Delta r_\rho + r_\rho \Delta r_c$$

$$\underbrace{\begin{bmatrix} \Delta s^1 \\ \Delta s^2 \\ \Delta s^3 \\ \Delta s^4 \\ \Delta s^5 \\ \Delta s^6 \\ \Delta s^7 \\ \Delta s^8 \end{bmatrix}}_{\Delta s^\mu} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ r^5 & 0 & 0 & 0 & r^1 & 0 & 0 & 0 \\ 0 & r^6 & 0 & 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & r^7 & 0 & 0 & 0 & r^3 & 0 \\ 0 & 0 & 0 & r^8 & 0 & 0 & 0 & r^4 \end{bmatrix}}_{t^\mu_\nu} \underbrace{\begin{bmatrix} \Delta r^1 \\ \Delta r^2 \\ \Delta r^3 \\ \Delta r^4 \\ \Delta r^5 \\ \Delta r^6 \\ \Delta r^7 \\ \Delta r^8 \end{bmatrix}}_{\Delta r^\nu}$$

t^μ_ν: from r to s, contravariant

Model re-parameterization as a coordinate transform

$$\begin{bmatrix} 1 & 0 & 0 & 0 & r^5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & r^6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & r^7 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & r^8 \\ 0 & 0 & 0 & 0 & r^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r^4 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ -1 \\ 0 \\ \omega^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega^2 r^7 - 2 \\ -1 \\ 0 \\ 0 \\ \omega^2 r^3 \\ 0 \end{bmatrix}$$

from s to r, covariant

$$\frac{\partial a_{33}}{\partial s^\mu} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \\ 0 \\ 0 \\ \omega^2 \\ 0 \end{bmatrix}$$

$$\frac{\partial a_{33}}{\partial r^\mu} = \begin{bmatrix} 0 \\ 0 \\ \omega^2 r^7 - 2 \\ -1 \\ 0 \\ 0 \\ \omega^2 r^3 \\ 0 \end{bmatrix}$$

✓

Transforms for efficient optimization

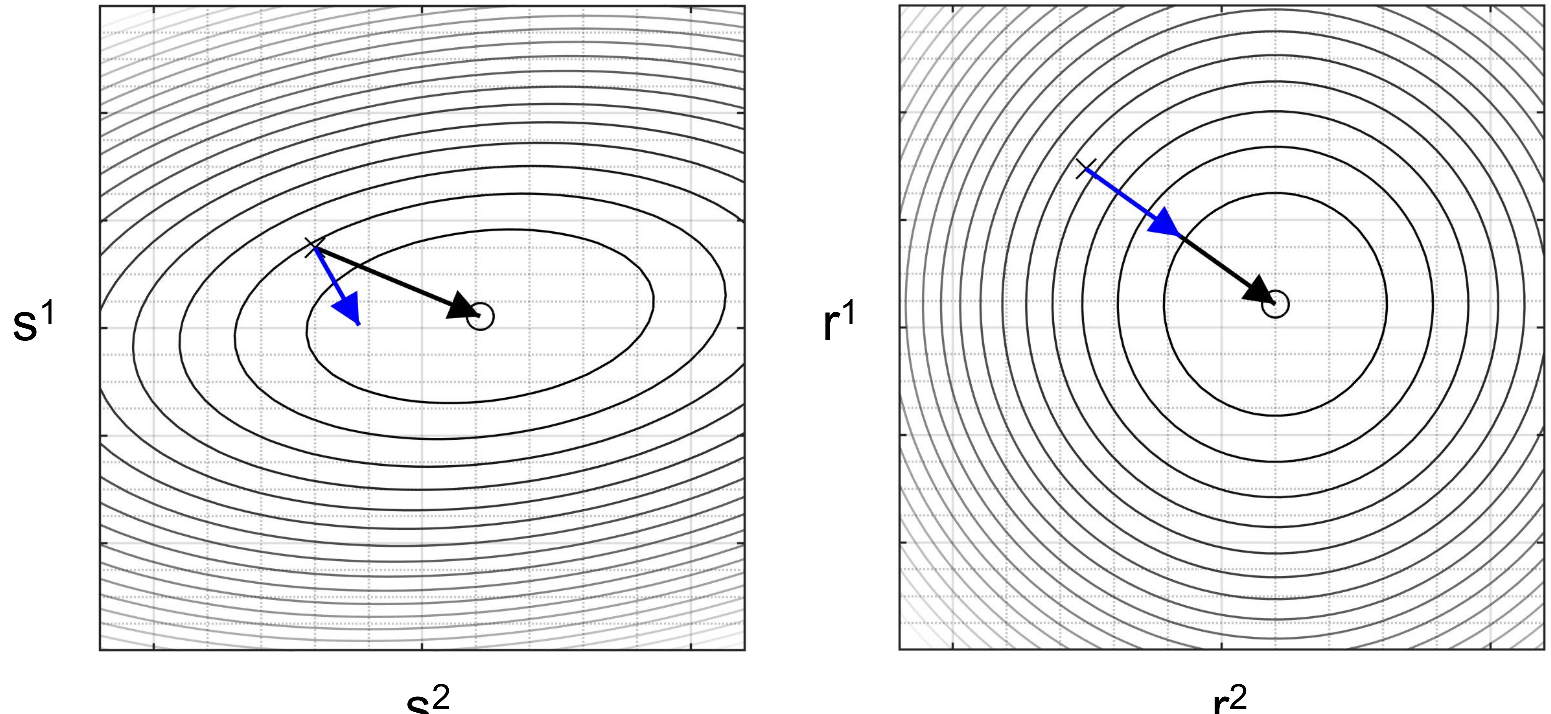
Motivation to seek alternative coordinate systems:

- Optimization, interpretation, regularization

Attempt one which leads to efficient optimization:

- 2nd derivative behaviour of the objective function determines uncertainties, cross-talk

$$\phi_{\mu\nu}(s) \neq \delta_{\mu\nu} \qquad \phi_{\mu\nu}(r) = \delta_{\mu\nu}$$



Transforms for efficient optimization

$$\text{Given } \Phi = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \phi_{1,3} & \cdots & \phi_{1,N} \\ \phi_{1,2} & \phi_{2,2} & \phi_{2,3} & \cdots & \phi_{2,N} \\ \phi_{1,3} & \phi_{2,3} & \phi_{3,3} & \cdots & \phi_{3,N} \\ & & \vdots & \ddots & \\ \phi_{1,N} & \phi_{2,N} & \phi_{3,N} & \cdots & \phi_{N,N} \end{bmatrix},$$

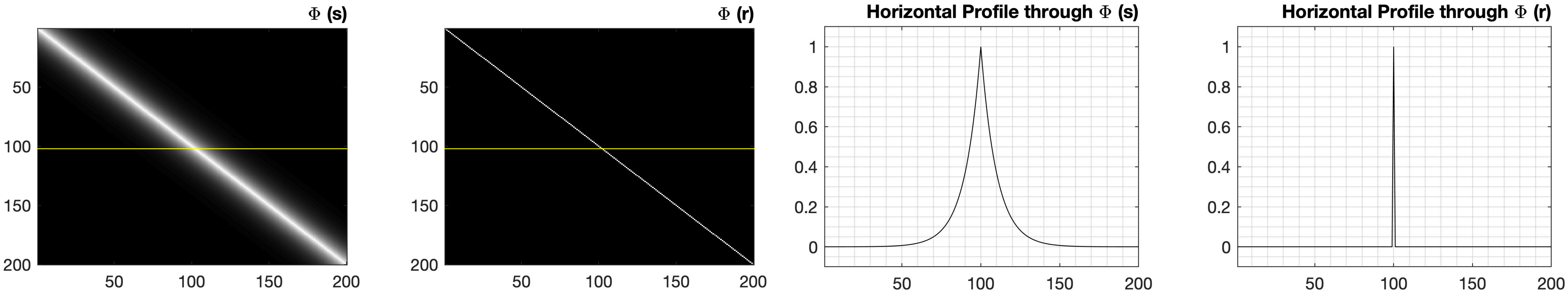
$$\text{find } \mathbf{T} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,N} \\ t^*_{2,1} & t_{2,2} & t_{2,3} & \cdots & t_{2,N} \\ t^*_{3,1} & t^*_{3,2} & t_{3,3} & \cdots & t_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \\ t^*_{N,1} & t^*_{N,2} & t^*_{N,3} & \cdots & t_{N,N} \end{bmatrix}$$

subject to $\mathbf{T}^T \Phi \mathbf{T} = \mathbf{I}$

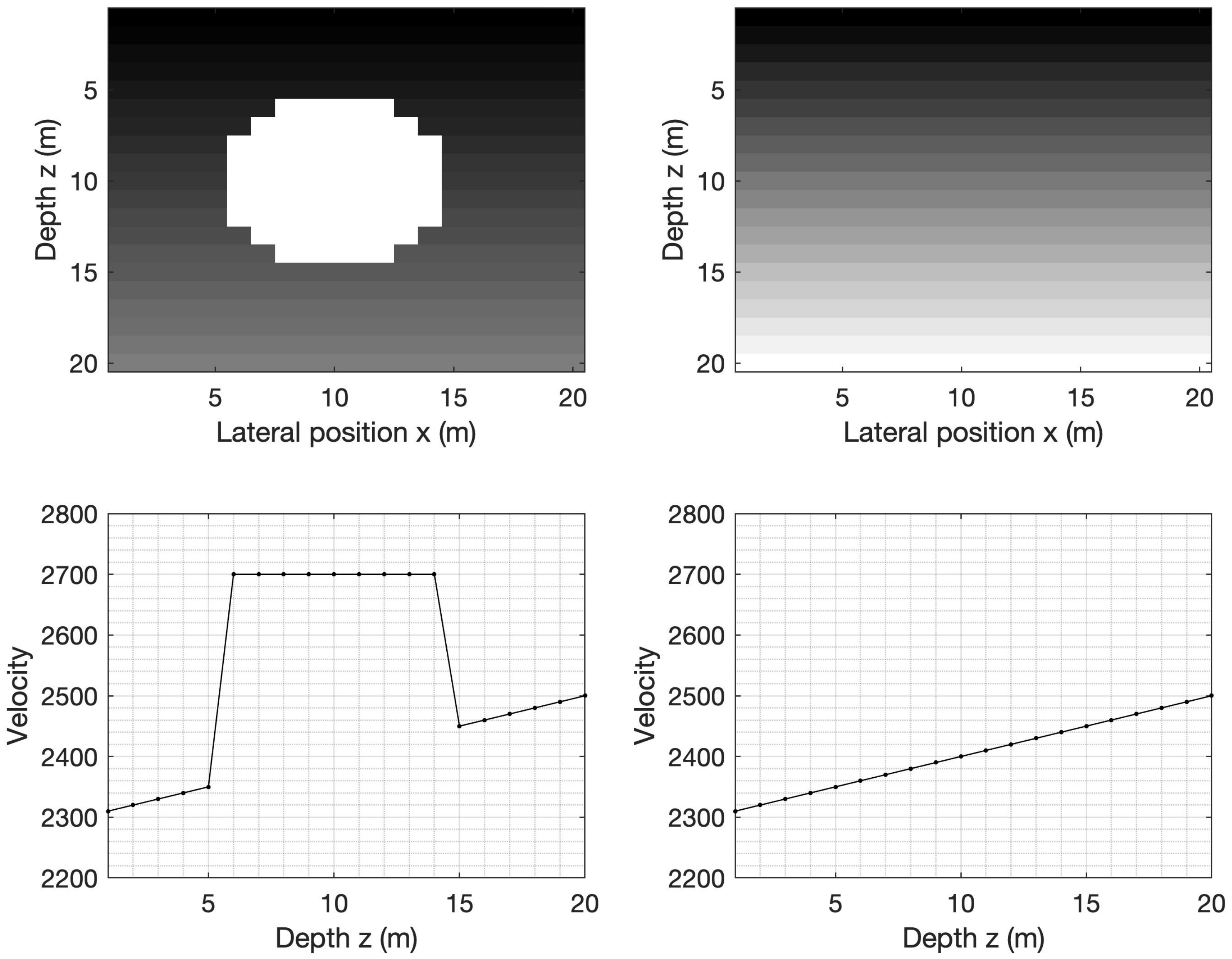
…with elements below the diagonal (marked *) preselected.

Numerics

- Algorithm to compute T column by column for any N available
- Pre-selected T entries can impact stability of algorithm
  - E.g., large entries can cause issues
  - Zeros and random entries with mean scaled in proportion to 1/N work well
  - “Smart choices” is still currently un-explored
- T is computed entirely through Hessian-vector products

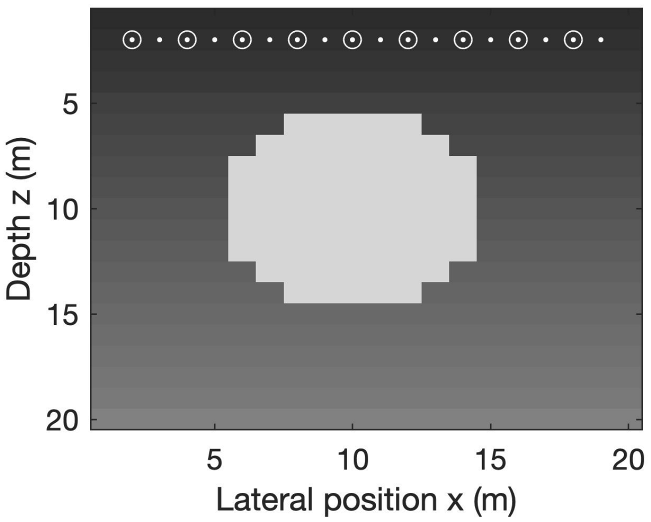


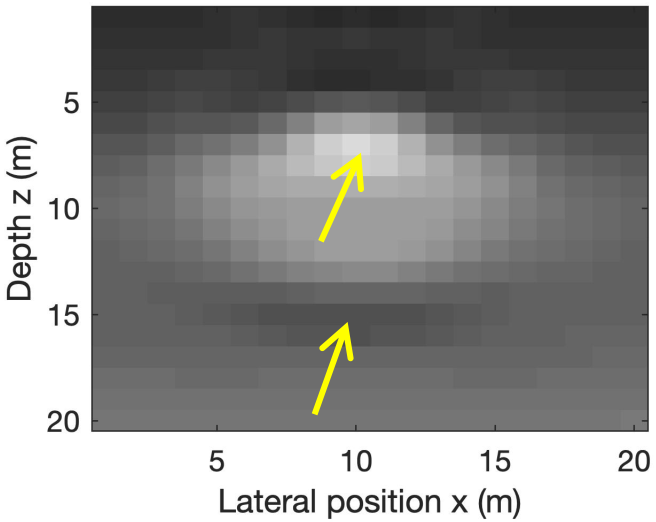
FWI example

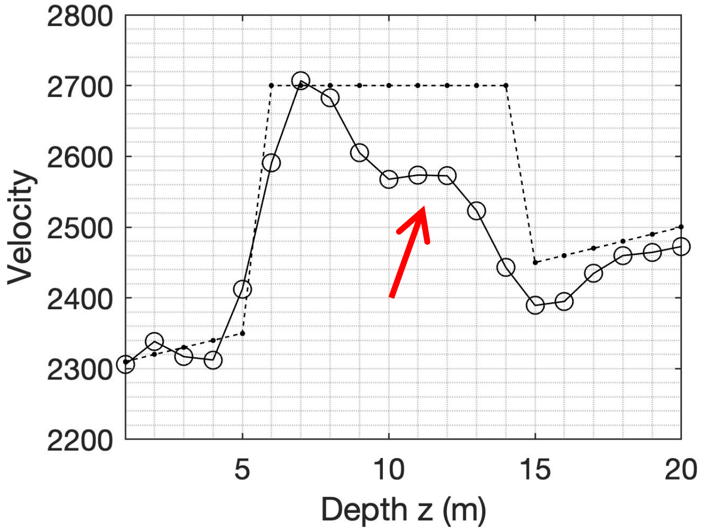


Depth z (m)
5
10
15
20
Lateral position x (m)
Velocity
2200
2300
2400
2500
2600
2700
2800

- Poor source/receiver coverage
- 20 iterations, multiscale
    - 5 frequencies per group
    - Evenly distributed from 1Hz:1Hz at iteration 1 to 1Hz:35Hz at iteration 20

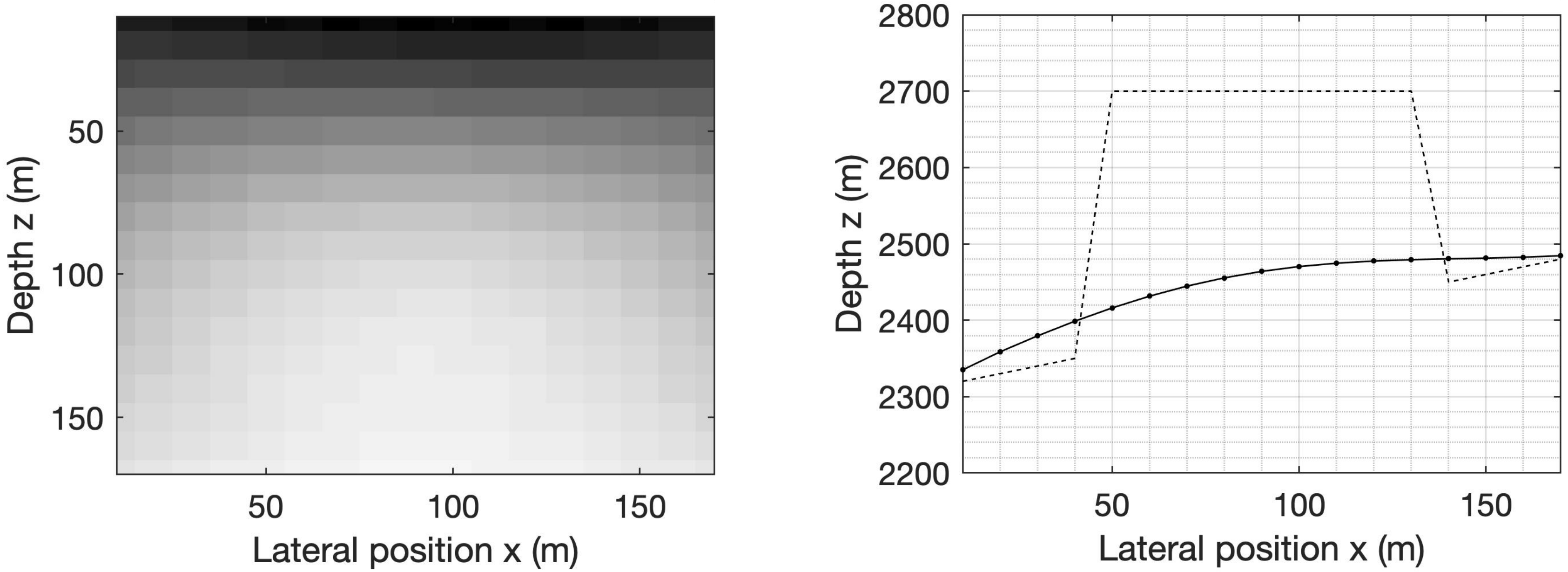






FWI example

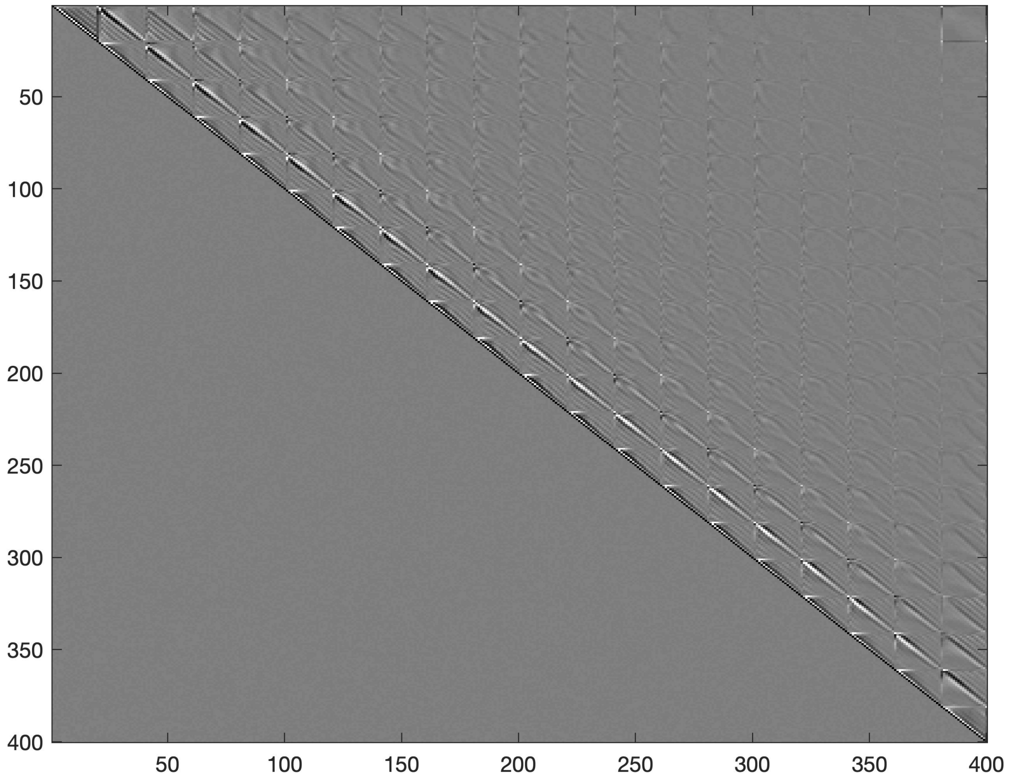
After iteration 4 is complete:



FWI example

$$\mathbf{T} = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,N} \\ t^*_{2,1} & t_{2,2} & t_{2,3} & \cdots & t_{2,N} \\ t^*_{3,1} & t^*_{3,2} & t_{3,3} & \cdots & t_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \\ t^*_{N,1} & t^*_{N,2} & t^*_{N,3} & \cdots & t_{N,N} \end{bmatrix}$$

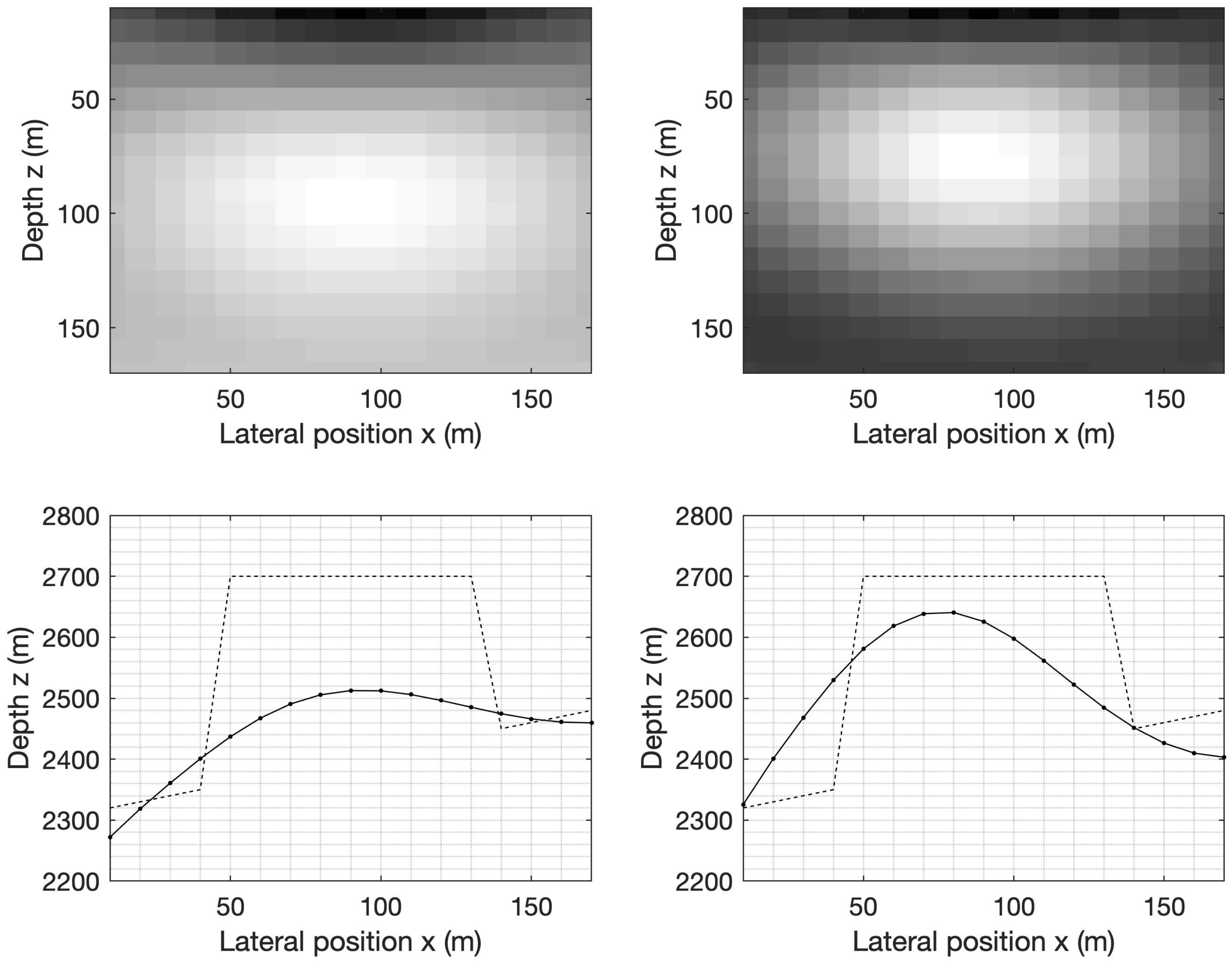
set lower triangular entries to zero



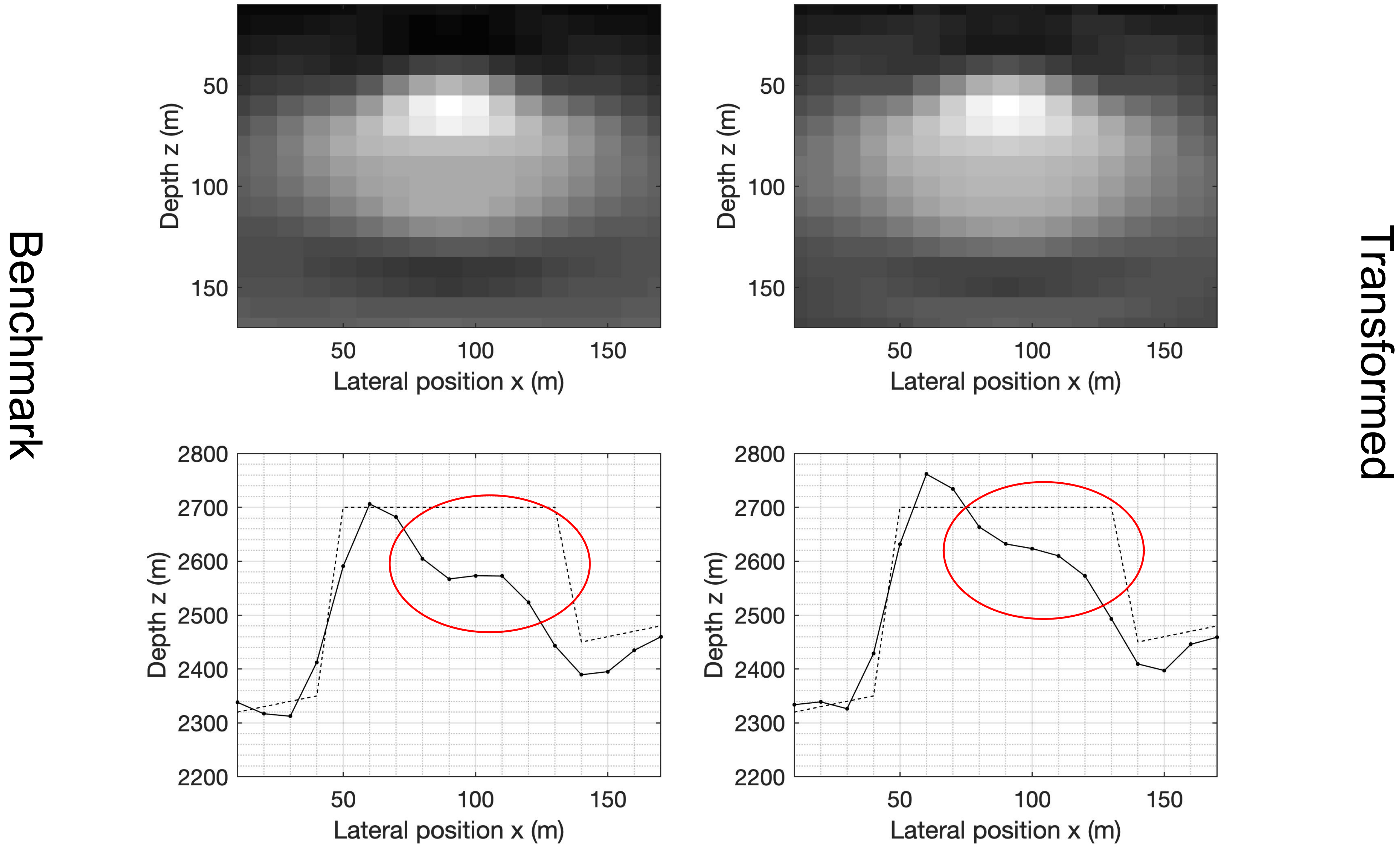
FWI example

IT 5 - benchmark

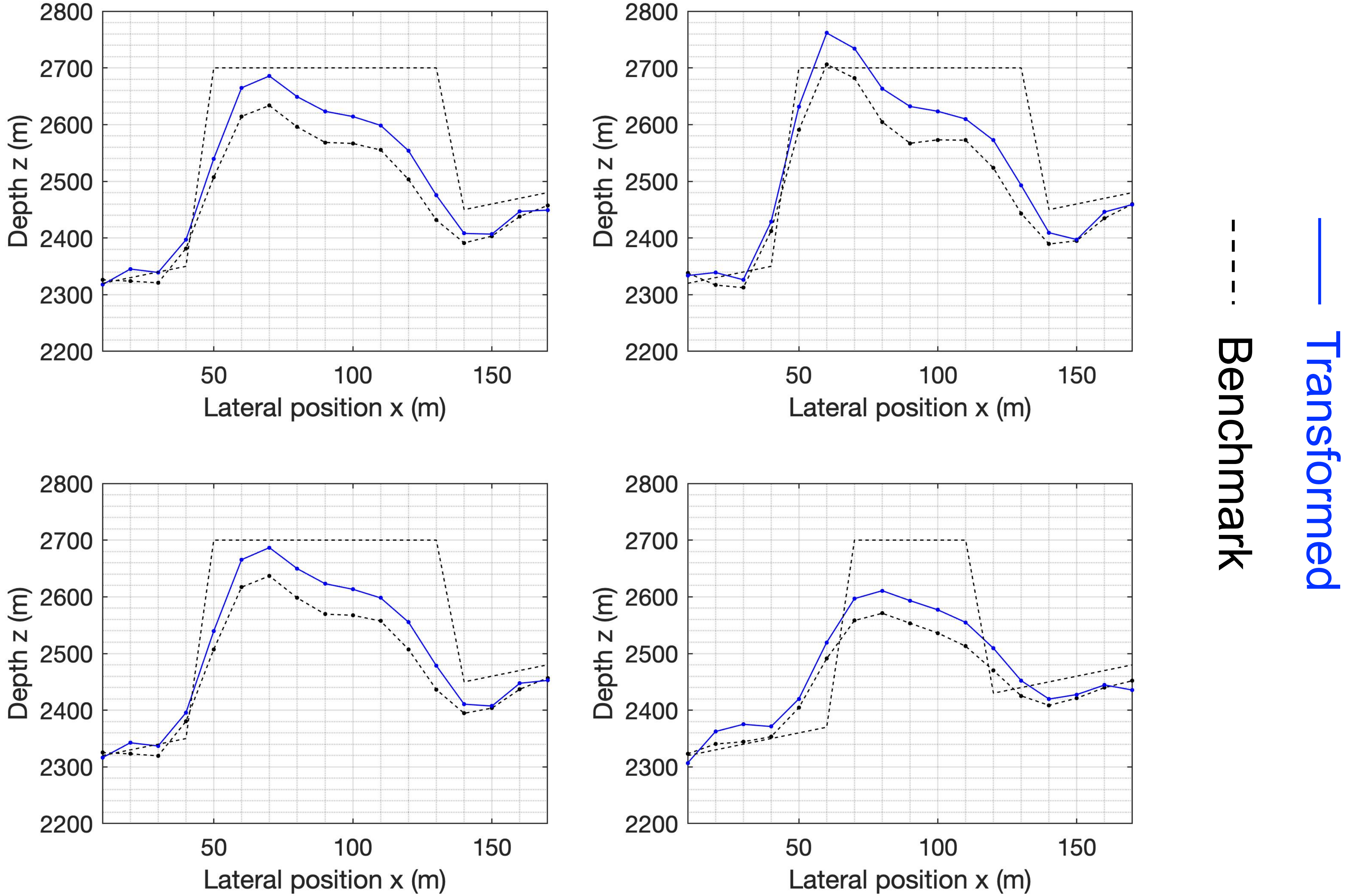
IT 5 - transformed



FWI example


Benchmark
Transformed
Depth z (m)
Lateral position x (m)
50
100
150
2800
2700
2600
2500
2400
2300
2200

FWI example



2800
2700
2600
2500
2400
2300
2200
Depth z (m)
50
100
150
Lateral position x (m)
Transformed
Benchmark

Conclusions

Re-parameterizations of inverse problems are transformations between rectilinear but non-Cartesian coordinate systems overlaying model space.

Motivates design of new transforms; for large problems, systems within which optimization is more efficient seem promising.

A class of transformation matrices T are available for which $H=I$

- Optimization in such systems should involve reduced parameter cross-talk
- Matrices computable through Hessian-vector products

In initial FWI examples

- Stable coordinate transforms are straightforwardly computed
- Steepest descent updates appear to compensate for illumination

Next

- Multiparameter FWI
- Invoking other considerations in T design – preconditioning?

Acknowledgments and further information

Thanks to:

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Brian Russell

More on tensors and vectors in non-Cartesian systems:

Spiegel, M., 1974, Theory and problems of vector analysis, Schaum's Outline Series

Dirac, P. A. M., 1975, General theory of relativity: Princeton University Press.

Borisenko, A., and I. Tarapov, 1968, Vector and tensor analysis with applications, Dover.